Antihydrogen Gravitational States

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Abstract

We present a theoretical study of the motion of the antihydrogen atom ($\bar{H}$) in the Earth's gravitational field above a material surface. We predict that $\bar{H}$ atom, falling in the Earth's gravitational field above a material surface, would settle in long-living quantum states. We point out a method of measuring the difference in energy of $\bar{H}$ in such states that allow us to apply spectroscopy of gravitational levels based on atom-interferometric principles. We analyze a general feasibility to perform experiments of this kind. We point out that such experiments provide a method of measuring the gravitational force (Mg) acting on $\bar{H}$ and they might be of interest in a context of testing the Weak Equivalence Principle for antimatter.

PACS numbers:

I. INTRODUCTION

Galileo, Newton and Einstein recognized that all bodies, regardless of their mass and composition, fall towards the Earth with an equal gravitational acceleration. Is that conclusion valid for antimatter? This has never been tested.

In the context of the general relativity theory, the universality of free fall is often referred to as the Weak Equivalence Principle (WEP). Violations of WEP could occur in ordinary matter-matter interactions e.g. as a result of the difference in the gravitational coupling to the rest mass and that to the binding energy. WEP is being tested with increasing sensitivity for macroscopic bodies. The best test so far confirms WEP to the accuracy of $2 \cdot 10^{-13}$ (using a rotating torsion balance [1]). Ongoing projects aim at the accuracy of 1 part in 10^{16} (laser tracking of a pair of test bodies in a freely falling rocket [2]), or even to 1 part in 10^{18} (in an Earth orbiting satellite [3]). However, in view of difficulties in unifying the quantum mechanics with the theory of gravity, it is of great interest to investigate the gravitational properties of *quantum mechanical objects*, such as elementary particles or atoms. Such experiments have been already performed, e.g. using interferometric methods to measure the gravitational acceleration of neutrons [4, 5] and atoms [6–9]. However, the experiments with *anti*atoms (see [10, 11] and references therein) are even more interesting in view of testing WEP, because the theories striving to unify gravity and quantum mechanics (such as supersymmetric string theories) tend to suggest violation of the gravitational equivalence of particles and antiparticles [12]. Experiments testing gravitational properties of antiatoms are on the agenda of all experimental groups working with antihydrogen (see e.g. ATHENA-ALPHA [13], ATRAP [14] and AEGIS [15]). One of the challenging aspects in experiments of this kind is to control the initial parameters of antiatoms, such as their temperature and position, with sufficient accuracy [16].

In the present paper we investigate a possibility to explore gravitational properties of antiatoms in the ultimate quantum limit. We study antihydrogen atoms levitating in the lowest gravitational states above a material surface. The existence of such gravitational states for *neutrons* was proven experimentally [17–19]. The existence of analogous states for antiatoms seems, at a first glance, impossible because of annihilation of antiatoms in the material walls. However, we have shown that ultracold antihydrogen atoms are efficiently reflected from material surface [20, 21] due to so-called quantum reflection from the Casimir-

Polder atom-surface interaction potential. We have shown that antihydrogen atoms, confined by the quantum reflection via Casimir forces from below, and by the gravitational force from above, would form metastable gravitational quantum states. They would bounce on a surface for a finite life-time (of the order of 0.1 s) [20]. This simple system can be considered as a microscopic laboratory for testing the gravitational interaction under extremely well specified (in fact, quantized!) conditions.

The annihilation of ultraslow antiatoms in a wall occurs with a small but finite (few percent) probability. It provides a clear and easy-to detect signal, which might be used to measure continuously the antiatom density in the gravitational states as a function of time. If antiatoms are settled in a superposition of gravitational states, the antiatom density evolves with beatings, determined by the *energy difference* between the gravitational levels. The transition frequencies between the gravitational levels are related to the strength of the gravitational force Mg, acting on antiatoms; here M is the gravitational mass of $\bar{H}$, and g is the Earth's local gravitational field strength. Also we show that a measurement of *differences* between the energy levels would allow us to disentangle Mg in a way independent on effects of the antiatom-surface interaction.

The plan of the paper is the following. In section II we study the main properties of the quasi-stationary gravitational states; in section III we present the time evolution of the $\bar{H}$ gravitational states superposition; in section IV we discuss a concept of a quantum ballistic experiment, namely the spatial-temporal evolution of the $\bar{H}$ gravitational states superposition, in section V we analyze the feasibility of measuring $\bar{H}$ atom properties in gravitational quantum states. In the Appendix we derive useful analytical expressions for the quasi-stationary gravitational states scalar product.

II. $\bar{H}$ GRAVITATIONAL STATES

In this section we discuss the properties of the $\bar{H}$ gravitational states above a material surface.

We consider an $\bar{H}$ atom bouncing above a material surface in the Earth's gravitational field. $\bar{H}$ is confined due to the quantum scattering from the Casimir-Polder potential below, and the gravitational field above. The Schrödinger equation for the $\bar{H}$ wave-function $\Psi(z)$

in such a superposition of atom-surface and gravitational potentials is:

$$\left[-\frac{\hbar^2 \partial^2}{2m \partial z^2} + V(z) + Mgz - E \right] \Psi(z) = 0 \tag{1}$$

Here z is the distance between the surface and the $\bar{H}$ atom, and $V(z)$ is the atom-surface interaction potential with a long-range asymptotic form $V(z) \sim -C_4/z^4$. We distinguish between the gravitational mass, that we refer to as M and the inertial mass, denoted by m hereafter. The wave-function $\Psi(z)$ satisfies the full absorption boundary condition at the surface ($z = 0$) [21], which stands for the annihilation of antiatoms in the material wall.

The characteristic length and energy scales are

$$l_0 = \sqrt[3]{\frac{\hbar^2}{2mMg}}, \tag{2}$$

$$l_{CP} = \sqrt{2mC_4}, \tag{3}$$

$$\varepsilon_0 = \sqrt[3]{\frac{\hbar^2 M^2 g^2}{2m}}, \tag{4}$$

$$\varepsilon_{CP} = \frac{\hbar^2}{4m^2 C_4}. \tag{5}$$

Here $l_0 = 5.871$ μm is the characteristic gravitational length scale, $l_{CP} = 0.027$ μm is the characteristic Casimir-Polder interaction length scale, $\varepsilon_0 = 2.211 \ 10^{-14}$ a.u. is the characteristic gravitational energy scale, and $\varepsilon_{CP} = 1.007 \ 10^{-9}$ a.u. is the Casimir-Polder energy scale. As one can see, the gravitational length scale is much larger than the Casimir-Polder length scale $l_0 \gg l_{CP}$, while the gravitational energy scale is much smaller than the Casimir-Polder energy scale $\varepsilon_0 \ll \varepsilon_{CP}$. It is useful to introduce the gravitational time scale τ_0:

$$\tau_0 = \hbar/\varepsilon_0 \simeq 0.001s \tag{6}$$

For large atom-surface separation distances $z \gg l_{CP}$ the solution of eq.(1) has a form:

$$\Psi(z) \sim \text{Ai}(\frac{z}{l_0} - \frac{E}{\varepsilon_0}) + K(E) \text{Bi}(\frac{z}{l_0} - \frac{E}{\varepsilon_0}) \tag{7}$$

where $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy functions [22]. The requirement of square integrability of the wave-function $\Psi(z \to \infty) \to 0$ results in the following equation for the energy levels of the gravitational states in the presence of the Casimir-Polder interaction:

$$K(E_n) = 0 \tag{8}$$

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n	λ_n^0	E_n^0, peV	z_n^0, μm
1	2.338	1.407	13.726
2	4.088	2.461	24.001
3	5.521	3.324	32.414
4	6.787	4.086	39.846
5	7.944	4.782	46.639
6	9.023	5.431	52.974
7	10.040	6.044	58.945

TABLE I: The eigenvalues, gravitational energies and classical turning points of a quantum bouncer with a mass of (anti)hydrogen in the Earth's gravitational field.

The hierarchy of the Casimir-Polder and gravitational scales $l_{CP} \ll l_0$ suggests that the quantum reflection from the Casimir-Polder potential can be accounted for by modifying the boundary condition for the quantum bouncer (a particle bouncing in the gravitational field above a surface, the interaction of the latter with a particle is modeled by infinite reflecting wall). The quantum bouncer wave-function satisfies the following equation system:

$$\begin{cases} \left[-\frac{\hbar^2 \partial^2}{2m \partial z^2} + Mgz - E_n \right] \Phi_n(z) = 0 \\ \Phi_n(z=0) = 0 \end{cases} \tag{9}$$

The quantum bouncer energy levels are known to be equal to [17]:

$$E_n^0 = \varepsilon_0 \lambda_n^0, \tag{10}$$

$$\mathrm{Ai}(-\lambda_n^0) = 0. \tag{11}$$

Table I summarizes the eigenvalues and classical turning points $z_n^0 = E_n^0/(Mg)$ for the first seven gravitational states of a quantum bouncer (with the mass of antihydrogen).

For the distances $l_{CP} \ll z \ll l_0$ one could neglect the gravitational potential in eq.(1). In this approximation, the solution of eq.(1) has the following asymptotic form:

$$\Psi(z) \sim \sin(kz + \delta(E)). \tag{12}$$

Here k is the wave vector $k = \sqrt{2mE}$, and $\delta(E)$ is the phase-shift of $\bar{H}$ reflected from the Casimir-Polder potential *in absence of the gravitational field* [21]. Matching asymptotics

in eq.(12) and eq.(7) we get a relation between the phase-shift $\delta(E)$ and the $K-$function introduced in Eq.(7):

$$K(E) = -\frac{\tan(\delta(E)) \, \mathrm{Ai}'(-E/\varepsilon_0) - kl_0 \, \mathrm{Ai}(-E/\varepsilon_0)}{\tan(\delta(E)) \, \mathrm{Bi}'(-E/\varepsilon_0) - kl_0 \, \mathrm{Bi}(-E/\varepsilon_0)}. \tag{13}$$

In deriving the above expression we took into account that the relation between $K(E)$ and $\delta(E)$ should not depend on the matching point z_m and thus can be formally attributed to $z_m = 0$. An equation for the distorted gravitational levels could be obtained by substitution of eq.(13) into eq.(8):

$$\frac{\tan(\delta(E_n))}{kl_0} = \frac{\mathrm{Ai}(-E_n/\varepsilon_0)}{\mathrm{Ai}'(-E_n/\varepsilon_0)}. \tag{14}$$

This equation is equivalent to the following boundary condition:

$$\frac{\Phi(0)}{\Phi'(0)} = \frac{\tan(\delta(E_n))}{k}. \tag{15}$$

Thus the following equation system describes an $\bar{H}$ atom, confined by the Earth's gravitational field and the quantum reflection from the Casimir-Polder potential:

$$\begin{cases} \left[-\frac{\hbar^2\partial^2}{2m\partial z^2} + Mgz - E_n\right]\Phi_n(z) = 0 \\ \frac{\Phi(0)}{\Phi'(0)} = \frac{\tan(\delta(E_n))}{k} \end{cases} \tag{16}$$

For the lowest gravitational states the condition $kl_{CP} \ll 1$ is valid. Thus the scattering length approximation for the phase-shift $\delta(E) \approx -ka_{CP}$ is well justified. The *complex-value* quantity [21]:

$$a_{CP} = -(0.10 + i1.05)l_{CP}, \tag{17}$$

$$a_{CP} = -0.0027 - i0.0287\mu m \tag{18}$$

is the scattering length on the Casimir-Polder potential provided full absorbtion in the material wall.

Thus the equation for the lowest eigenvalues (14) has a form:

$$\frac{a_{CP}}{l_0} = -\frac{\mathrm{Ai}(-E_n/\varepsilon_0)}{\mathrm{Ai}'(-E_n/\varepsilon_0)}. \tag{19}$$

The above equation is equivalent to the following boundary condition for the wave-function $\Phi(z)$ of a particle in the gravitational potential eq.(1):

$$\Phi(z \to 0) \to z - a_{CP} \tag{20}$$

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Because of the imaginary part of the scattering length a_{CP}, the gravitational states of $\bar{H}$ above a material surface are *quasi-stationary decaying* states. For low quantum numbers n, it is easy to relate the lowest quasi-stationary energy levels E_n to the unperturbed gravitational energy levels E_n^0 of a quantum bouncer. Indeed the variable substitution $z = \widetilde{z} + a_{CP}$ transforms Eq.(9,20) to the equation system for the quantum bouncer:

$$\left[-\frac{\hbar^2 \partial^2}{2m \partial \widetilde{z}^2} + Mg\widetilde{z} - (E_n - Mga_{CP}) \right] \Phi_n(\widetilde{z}) = 0 \tag{21}$$

$$\Phi_n(\widetilde{z} \to 0) \to 0 \tag{22}$$

The eigenvalues E_n and eigenfunctions Φ_n are :

$$E_n = E_n^0 + Mga_{CP}, \tag{23}$$

$$\Phi_n(z) = \frac{1}{N_i} \, \mathrm{Ai}((z - a_{CP})/l_0 - \lambda_n^0), \tag{24}$$

where N_i is the normalization coefficient (see Eqs.(78,79) in the Appendix). In the following, we will use the dimensionless eigenvalues $\lambda_n = E_n/\varepsilon_0$:

$$\lambda_n = \lambda_n^0 + a_{CP}/l_0 \tag{25}$$

An important message from the above expression is that the complex shift Mga_{CP} (due to the account of quantum reflection on the Casimir-Polder potential) is *the same* for all low-lying quasi-stationary gravitational levels. It means that the transition frequencies between the gravitational states are not affected by the Casimir-Polder interaction, provided the latter can be described by the complex scattering length a_{CP}. The scattering approximation is valid in the limit $k_n a_{CP} \to 0$, where $k_n = \sqrt{2mE_n}$ (let us note that for the first gravitational state $|k_1 a_{CP}| = 0.0071$). However, accounting for the higher order k-dependent terms in Eq.(14) would result in the state dependent shift of the gravitational states due to the Casimir-Polder interaction. We use a known low energy expansion of the s-wave phase-shift $\delta(E)$ in a homogeneous $1/z^4$ potential [23], in which we keep the two leading k-dependent terms:

$$a_{CP}k \cot(\delta(k)) \simeq -1 + \frac{\pi}{3}\frac{l_{CP}}{a_{CP}}(l_{CP}k) + \frac{4}{3}(l_{CP}k)^2 \ln\left(\frac{l_{CP}k}{4}\right) + ... \tag{26}$$

We introduce a k-dependent modified "scattering length" $\tilde{a}_{CP}(k) \equiv -\delta(k)/k$ and get the following expression for $\tilde{a}_{CP}(k)$:

$$\tilde{a}_{CP}(k) \simeq a_{CP} + \frac{\pi}{3} l_{CP}(l_{CP}k) + \frac{4}{3} a_{CP}(l_{CP}k)^2 \ln \frac{l_{CP}k}{4} \tag{27}$$

The leading k-dependent term in the above expression $\frac{\pi}{3}l_{CP}(l_{CP}k)$ is real and independent on properties of the inner part of the Casimir-Polder interaction. It is determined by the asymptotic form of the potential, thus it depends on the Casimir-Polder length scale l_{CP} only. Then the modified equation for the gravitational state energies is:

$$E_n = E_n^0 + Mg\tilde{a}_{CP}(E_n) \tag{28}$$

Taking into account the smallness of the k-dependent terms (for the lowest gravitational states) in expression (27), we get:

$$E_n \simeq E_n^0 + Mg\tilde{a}_{CP}(k_n^0) = \varepsilon\left(\lambda_n^0 + a_{CP}/l_0 + \frac{\pi l_{CP}}{3l_0}(l_{CP}k_n^0) + \frac{4a_{CP}}{3l_0}(l_{CP}k_n^0)^2\ln\frac{l_{CP}k_n^0}{4}\right). \tag{29}$$

Here $k_n^0 = \sqrt{2mE_n^0}$.

The account for k-dependent terms in Eq.(27) modifies the transition frequencies between the gravitational states in a way, dependent on the Casimir-Polder interaction. However, such modification is very weak. Indeed, taking into account, that $l_{CP}k_n^0 \sim l_{CP}/l_0$ for the lowest gravitational states , the leading k-dependent term corrections to the gravitational energy are of the second order in a small parameter l_{CP}/l_0. The transition frequency between the first and second gravitational states equals $\omega_{12} = \omega_{12}^0 + \Delta_{12}$, where $\omega_{12}^0 = (E_2^0 - E_1^0)/(2\pi\hbar) = 254.54$ Hz, and $\Delta_{12} = Mg(\tilde{a}_{CP}(k_2^0) - \tilde{a}_{CP}(k_1^0)) = 0.0017$ Hz.

An account of the first two terms in Eq.(27) provides equal decay width for the lowest gravitational states. This width is determined by the probability of antihydrogen penetrating to the surface and annihilating.

$$\Gamma_n = \varepsilon\frac{b}{2l_0}. \tag{30}$$

Here we use a standard notation $b = 4\,\mathrm{Im}\,a_{CP}$:

$$b = 0.115\ \mu m. \tag{31}$$

The widths of the gravitational states (30) are proportional to the ratio ε/l_0. Using Eqs. (2) and (4) we could find that this ratio is equal to the gravitational force $\varepsilon/l_0 = Mg$ so that

$$\Gamma_n = \frac{b}{2}Mg. \tag{32}$$

The corresponding life-time (calculated for an ideal conducting surface) is

$$\tau = \frac{2\hbar}{Mgb} \simeq 0.1 \text{ s.} \tag{33}$$

We note factorization of the gravitational effect (appearing in the above formula via a factor Mg) and the quantum reflection effect, manifestating through the constant b. Such a factorization is a consequence of the smallness of the ratio of the characteristic scales $b/(2l_0) \simeq 0.01$.

Comparing the $\bar{H}$ lifetime in the lowest gravitational states with the classical period $T = 2\sqrt{\frac{2l_0\lambda_1}{g}} \simeq 0.0033$ s of $\bar{H}$ with the ground state energy bouncing in the gravitational field, we see that $\bar{H}$ bounces in average about 30 times before annihilating. This shows that the lowest gravitational states are well resolved quasi-stationary states.

It is interesting to estimate the maximum gravitational quantum number N, below which the gravitational states are still resolved, i.e:

$$\frac{\tau_N}{T_N} = \frac{2\pi\hbar}{\Gamma(N)\frac{dE(n)}{dn}} > 1. \tag{34}$$

Here τ_N is the lifetime of the $N-$th gravitational state, T_N is a classical period, corresponding to the $N-$th state via $T_N = 2\hbar\pi/(dE(n)/dn)$. For such an estimation we transform eq.(14) using the asymptotic form of the Airy function for a large negative argument and get:

$$\lambda_n = \left(\frac{3}{2}(\pi[n - \frac{1}{4}] - \delta(E_n))\right)^{2/3}. \tag{35}$$

The accuracy of the above equation increases with increasing n; it gives the energy value within a few percent even for $n = 1$. In the energy domain of interest $|\delta(E)| \ll \pi[n - \frac{1}{4}]$ [21], so:

$$\lambda_n \simeq \lambda_n^0 - \frac{\delta(E_n^0)}{\sqrt{\lambda_n^0}}. \tag{36}$$

Here we used the semiclassical approximation for λ_n^0 [24]:

$$\lambda_n^0 \simeq \left(\frac{3}{2}\pi[n - \frac{1}{4}]\right)^{2/3} \tag{37}$$

One could verify that in the case of small n the above equation reduces to eq.(25). Substitution of eq.(36) into eq.(34) results to:

$$\frac{\tau_n}{T_n} \simeq \frac{1}{4\,\text{Im}\,\delta(E_n^0)}. \tag{38}$$
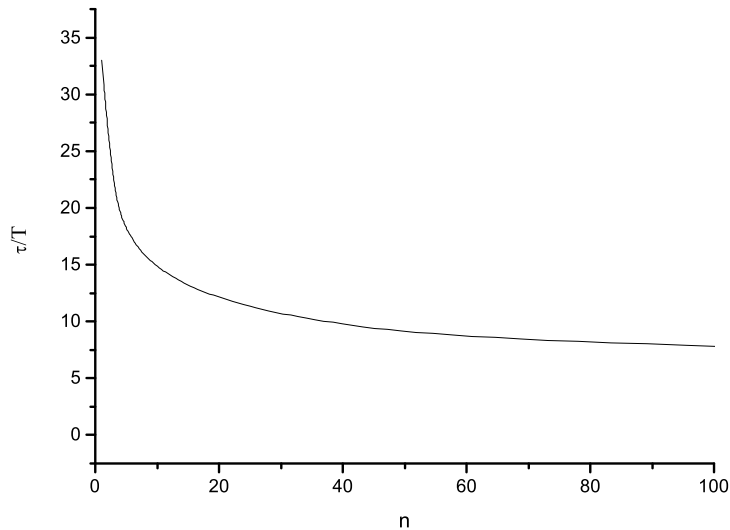
FIG. 1: The number of $\bar{H}$ bounces during the lifetime of $n-$th gravitational state

The ratio $\tau(n)/T(n)$ expresses the number of $\bar{H}$ classical bounces during the lifetime of the n-th state. This dependence is shown in Fig.1. Using numerical values $\delta(E)$, calculated in [21], we find that inequality (34) holds for

$$n < N = 30000. \tag{39}$$

The corresponding energy $E_N = 6 \ 10^{-11}$ a.u., and the characteristic size of such states is as large as $H_N = 1.6 \ cm$. This means that the concept of the quasi-bound gravitational states is justified not only for the lowest states, but it also might be applied for highly excited states.

The quasi-stationary character of the antiatom gravitational states above a material surface manifests itself in a nonzero current through the bottom surface ($z = 0$). Indeed, the expression for the current is:

$$j(z,t) = \frac{i\hbar}{2M} \left(\Phi(z,t)\frac{d\Phi^*(z,t)}{dz} - \Phi^*(z,t)\frac{d\Phi(z,t)}{dz} \right). \tag{40}$$

taken at $z = 0$ for a given gravitational state (24) turns out to be equal to:

$$j(0,t) = \varepsilon \exp(-\frac{\Gamma}{\hbar}t)\frac{\mathrm{Ai}^*(-\lambda_n)\,\mathrm{Ai}'(-\lambda_n) - \mathrm{Ai}(-\lambda_n)\,\mathrm{Ai}'^*(-\lambda_n)}{N_i N_i^*}. \tag{41}$$

Here λ_n is given by Eq. (25), N_i is the normalization factor. We take into account the smallness of the ratio a_{CP}/l_0 and Eq.(11), and get :

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$$\text{Ai}(-\lambda_n) \approx -\frac{a_{CP}}{l_0} \text{Ai}'(-\lambda_n^0), \tag{42}$$

which is exact up to the second order in the ratio a_{CP}/l_0. Now taking into account an explicit form of the normalization coefficients (Eq.(79) in Appendix) $N_i = \text{Ai}'((-\lambda_n^0)$, we get finally:

$$j(0,t) = -\varepsilon \frac{b}{2\hbar l_0} \exp(-\frac{\Gamma}{\hbar}t) = -\frac{\Gamma}{\hbar} \exp(-\frac{\Gamma}{\hbar}t). \tag{43}$$

This result is in full agreement with Eq.(30) as far as:

$$\frac{d}{dt} \int_0^\infty |\Phi(z,t)|^2 dz = j(0,t) = -\frac{\Gamma}{\hbar} \exp(-\frac{\Gamma}{\hbar}t). \tag{44}$$

III. BOUNCING ANTIHYDROGEN

In this section, we are interested in the evolution of an initially prepared arbitrary superposition of several lowest gravitational states of $\bar{H}$. In the following, we will limit our treatment to the scattering length approximation for describing the Casimir-Polder interaction, and will neglect all, except the first, term in the expression (27), so that $\tilde{a}_{CP}(k) \approx a_{CP}$. The corresponding $\bar{H}$ wave-function is:

$$\Phi(z,t) = \sum_{i=1}^n \frac{C_i}{N_i} \text{Ai}(z/l_0 - \lambda_i) \exp(-i\lambda_i \frac{t}{\tau_0}). \tag{45}$$

Here τ_0 is the characteristic $\bar{H}$ bouncer time scale, C_i are expansion coefficients and $N_i = \text{Ai}'(-\lambda_i)$ are the normalization factors of the gravitational states (see the Appendix).

We are interested in the evolution of the number of antihydrogen atoms as a function of time:

$$F(t) = \int_0^\infty |\Phi(z,t)|^2 dz = \sum_{i,j=1}^n \int_0^\infty \frac{C_j^* C_i}{N_j^* N_i} \text{Ai}^*(z/l_0 - \lambda_j) \text{Ai}(z/l_0 - \lambda_i) \exp(-i\varepsilon(\lambda_i - \lambda_j^*)t)dz \tag{46}$$

First, let us note that the above expression for the total number of particles is no longer constant because of the decay of the quasi-stationary gravitation states. Second, the quasi-stationary gravitational states corresponding to different energies are non-orthogonal:

$$\frac{1}{N_i N_j} \int_0^\infty \text{Ai}^*(z/l_0 - \lambda_j) \text{Ai}(z/l_0 - \lambda_i)dz \equiv \alpha_{ij} \neq \delta_{ij}. \tag{47}$$

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In the Appendix we will derive the following expression for the cross-terms α_{ij}, exact up to the second order of the small ratio a_{CP}/l_0:

$$\alpha_{i \neq j} = i \frac{b/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)} \tag{48}$$

As one can see, such cross-terms vanish if there is no decay, i.e. if $b = 4 \operatorname{Im} a_{CP} \to 0$.

Now we can calculate an expression for the number of antihydrogen atoms as a function of time (46):

$$F(t) = \exp(-\frac{\Gamma}{\hbar}t) \left(\sum_i^n |C_i|^2 + 2 \operatorname{Re} \sum_{i>j}^n \sum_j^n C_j^* C_i \frac{ib/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)} \exp(-i(\lambda_i^0 - \lambda_j^0)\frac{t}{\tau_0}) \right). \tag{49}$$

From Eqs.(49) and (30) we get the following expression for the disappearance (annihilation) rate $-\frac{dF(t)}{dt}$, keeping the terms up to the second order in the ratio a_{CP}/l_0:

$$\frac{dF(t)}{dt} = -\frac{\Gamma}{\hbar} \exp(-\frac{\Gamma}{\hbar}t) \left(\sum_i^n |C_i|^2 + 2 \operatorname{Re} \sum_{i>j}^n \sum_j^n C_j^* C_i \exp(-i(\lambda_i^0 - \lambda_j^0)\frac{t}{\tau_0}) \right). \tag{50}$$

For a superposition of the two gravitational states with the equal coefficients $C_{1,2}$ (say $C_1 = C_2 = 1$), the above expression gets a simple form:

$$\frac{dF_{12}(t)}{dt} = -\frac{\Gamma}{\hbar} \exp(-\frac{\Gamma}{\hbar}t) \left(1 + \cos(\omega_{12}t)\right). \tag{51}$$

Here $\omega_{12} = (\lambda_2^0 - \lambda_1^0)/\tau_0$. The same result could be obtained by calculating the flux $j(0, t)$ Eq.(40) for a superposition of states (45).

One can see, that the disappearance rate decays as a function of time according to the exponential law with the width Γ (the same for the lowest states), also it oscillates with the transition frequency between the first and second gravitational states (equal to 254.54 Hz). We plot in Fig.2 the time evolution of $\bar{H}$ disappearance rate in a superposition of two lowest states. Curiously, the oscillation of disappearance rate is the direct consequence of decaying character of gravitational states. Indeed, such an oscillation is observable due to the nonvanishing contribution of the interference term in the expression for the total probability to find antihydrogen atoms, given by Eq.(49). As one can see from Eq.(48) this contribution is proportional to the imaginary part of the scattering length, it would vanish in case there were no decay of gravitational states due to annihilation in the material wall, described by parameter b/l_0.
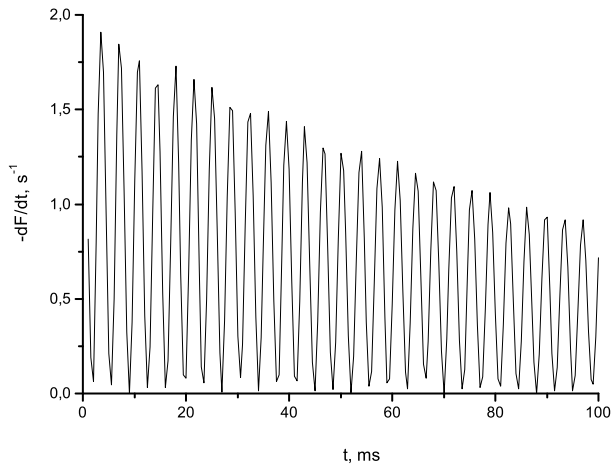
FIG. 2: Evolution of the annihilation rate of $\bar{H}$ atom in a superposition of the first and the second gravitational states.
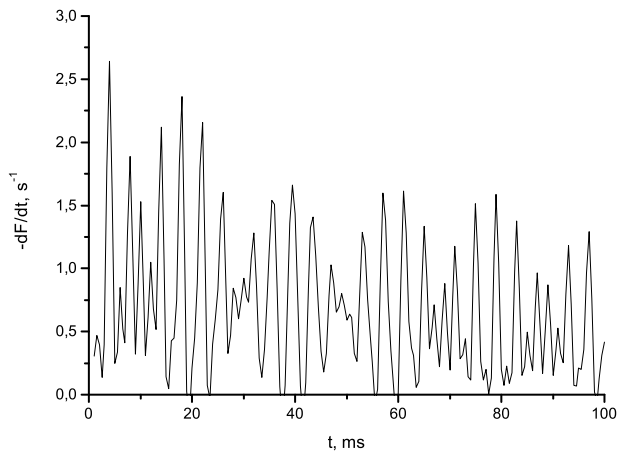

FIG. 3: Evolution of the annihilation rate of $\bar{H}$ atom in a superposition of first, second and third gravitational states.

So far the oscillation frequency of the disappearance rate $N(t)$ corresponds to the energy difference between the unperturbed gravitational levels. Expression (50) does not include the shift of gravitational state energies $\mathrm{Re}\, a_{CP}/l_0$ as it is equal for all the gravitational states, thus it is canceled out in the energy difference. The account for higher order k-dependent terms in (27) would result in a small (second order of the ratio (a_{CP}/l_0)) correction to the transition frequency. A measurement of the oscillation frequency ω_{12} given by Eq. (51)

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would allow us to extract the following combination of the gravitational and the inertial masses from eq.(4):

$$\frac{M^2}{m} = \frac{2\hbar\omega_{12}^3}{g^2(\lambda_2^0 - \lambda_1^0)^3}.$$

(52)

Under the additional assumption of the equality of the known inertial mass of the *hydrogen* atom m_H and that of antihydrogen, imposed by CPT, we get:

$$M = \sqrt{\frac{2m_H\hbar\omega_{12}^3}{g^2(\lambda_2^0 - \lambda_1^0)^3}}.$$

(53)

The evolution of *three* gravitational state superposition provides information not only about the characteristic energy scale ε_0 but also about the level spacing as a function of quantum number n, characterized by the value $d^2E(n)/dn^2$. Such a study might be interesting for testing additional (to Newtonian gravitation) interactions (see [25, 26] and references there in) between $\bar{H}$ and a material surface with the characteristic spatial scale of the order of micrometers. Such interactions would manifest as nonlinear additions to the gravitational potential, which would modify the spectrum character. In the case of three state superposition, the disappearance rate (50) has the form:

$$\frac{dF_{123}(t)}{dt} = -\frac{2}{3}\frac{\Gamma}{\hbar}\exp(-\frac{\Gamma}{\hbar}t)\left(\frac{3}{2} + \cos(\omega_{12}t) + \cos(\omega_{23}t) + \cos((\omega_{12} + \omega_{23})t)\right).$$

(54)

Here $\omega_{ij} = (\lambda_j^0 - \lambda_i^0)/\tau_0$. One could verify that the period of coherence of $\cos(\omega_{12}t)$ and $\cos(\omega_{23}t)$ terms is:

$$T_r = \frac{2\pi}{\omega_{12} - \omega_{23}} \simeq 0.02s.$$

(55)

A semiclassical expression for T_r is:

$$T_r \approx \frac{2\pi}{|d^2E/dn^2|}.$$

(56)

One can see that the period T_r is a quantum limit analog of a half revival period $T_{rev} = 4\pi/|d^2E/dn^2|$ (T_{rev} characterizes the time period after which the evolution of the wavepacket returns to the semiclassical behavior, see [27] for details and reference therein). In Fig.3 we plot the annihilation events as a function of time (54) for a superposition of three lowest gravitational states. The period T_r is clearly seen as a period of modulation of a rapidly oscillating function. The ratio

$$T_r/\tau_0 = \frac{2\pi}{\lambda_3 - 2\lambda_2 + \lambda_1}$$

(57)

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is sensitive to any nonlinear addition to the gravitational potential. Indeed while linear corrections to gravitational potential can only change ε_0, nonlinear additions change the derivative of levels density $|d^2E/dn^2|$.

IV.   QUANTUM BALLISTIC EXPERIMENT

Two independent experiments are needed in order to determine the gravitational mass M and the inertial mass m of antihydrogen. In the previous section, we showed that a combination of gravitational and inertial masses M and m, given by Eq.(52), can be extracted from the frequency measurement eq.(51, 54). An independent information could be obtained from measurement of the spatial density distribution of $\bar{H}$ in a superposition of the gravitation states, for instance, in the flow-throw experiment (a kind of a beam scattering experiment), in which $\bar{H}$-atoms with a wide horizontal velocity distribution move along the mirror surface. The time of flight along the mirror should be measured simultaneously with the spatial density distribution in a position-sensitive detector, placed at the mirror exit. Such a detector would be able to measure the density distribution along the vertical axis at a given time instant. The horizontal component of $\bar{H}$ motion could be treated classically. Due to a broad distribution of horizontal velocities in the beam, atoms would be detected within a wide range of time intervals between their entrance to the mirror and their detection at the exit. In such an approach, we could study the time evolution of $\bar{H}$ probability density at a given position z:

$$|\Phi_{(12)}(z,t)|^2 = \exp(-\frac{\Gamma}{\hbar}t)\left(|\Phi_{(12)}^{av}(z)|^2 + 2\operatorname{Re}\frac{\operatorname{Ai}(z/l_0 - \lambda_1)\operatorname{Ai}(z/l_0 - \lambda_2)}{\operatorname{Ai}'(-\lambda_1)\operatorname{Ai}'(-\lambda_2)}\exp(-i\omega_{12}t)\right),\text{(58)}$$

$$|\Phi_{(12)}^{av}(z)|^2 = \left|\frac{\operatorname{Ai}(z/l_0 - \lambda_1)}{\operatorname{Ai}'(-\lambda_1)}\right|^2 + \left|\frac{\operatorname{Ai}(z/l_0 - \lambda_2)}{\operatorname{Ai}'(-\lambda_2)}\right|^2 \text{(59)}$$

The transition $\omega_{12} = 254.54 Hz$ could be extracted from the probability density time evolution at a given z. The length scale l_0 could be extracted from the position of the zero $z_1^{(2)}$ of the wave-function. Here superscript stands for the state number, and a subscript corresponds to the number of zero for a given state. Thus such a position is determined by the condition $\operatorname{Ai}(z_1^{(2)}/l_0 - \lambda_2) = 0$; is equal to the following expression:

$$z_1^{(2)} = (\lambda_2 - \lambda_1)l_0 = 10.27\mu m. \tag{60}$$

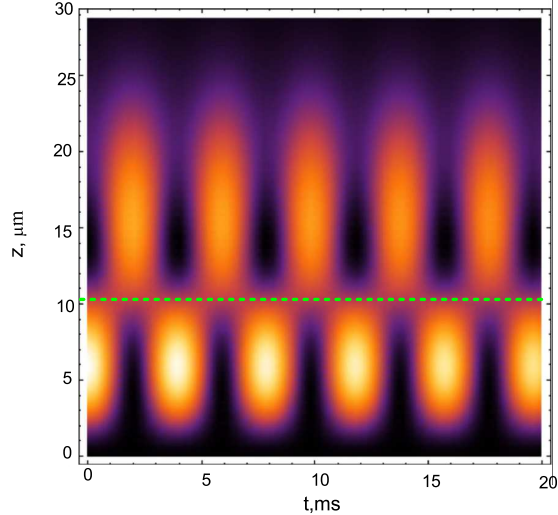The probability density in Eq.(58) of a two states superposition at $z = z_1^{(2)}$ behaves like the

FIG. 4: Color.The probability density of $\bar{H}$ in a superposition of the first and second gravitational states, as a function of the height z above a mirror (vertical-axis) and the time t (horizontal-axis). Dark shade: low probability density. Light shade: high probability density. The dashed line indicates the position of the node in the wave-function of the second state

probability density of the ground state alone:

$$|\Phi_{12}(z_1^{(2)}, t)|^2 = \exp(-\frac{\Gamma}{\hbar}t)\left|\frac{\text{Ai}(z_1^{(2)}/l_0 - \lambda_1)}{\text{Ai}'(-\lambda_1)}\right|^2. \tag{61}$$

The probability density at a height $z_1^{(2)}$ does not exhibit any time-dependent oscillations. We show the probability density as a function of the height z above a mirror (y-axis) and the time t (x-axis) in a superposition of the first and second gravitational states in Fig.4.

The position of the node in the wave-function of the second state is shown in Fig.4 as a horizontal line, separating lower and upper rows of periodic maxima and minima in the probability density plot. The position $z_1^{(2)}$ does not depend on initial populations of the gravitational states, which makes it beneficial for extracting the spatial scale l_0.

From knowing the length l_0 and the energy ε_0 scales, one could get the following expressions for the inertial m and gravitational M masses of $\bar{H}$:

$$m = \frac{\hbar^2}{2\varepsilon_0 l_0^2}, \tag{62}$$

$$M = \frac{\varepsilon_0}{g l_0}. \tag{63}$$

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The equality $m = M$ postulated by EP relates ε_0 and l_0 as follows:

$$\varepsilon_0 = \hbar\sqrt{\frac{g}{2l_0}}, \tag{64}$$

or, using the gravitational time scale Eq.(6):

$$\tau_0 = \sqrt{\frac{2l_0}{g}}. \tag{65}$$

One can easily recognize in the above expression a classical time of fall from the height l_0 in the Earth's gravitational field.

Thus a measurement of the temporal-spatial probability density dependence of $\bar{H}$ in a superposition of the two lowest gravitation states would provide a full information on the gravitational properties of antimatter. The superposition of three (and more) gravitational states could be useful to search for additional (to gravity) interactions with a spatial scale of the order of l_0. For such a purpose, it is useful to study the probability density at zeros of each Airy function in the state superposition. In particular, the positions corresponding to zeros of second and third gravitational states are the following:

$$z_1^{(2)} = (\lambda_2 - \lambda_1)l_0 = 10.27\mu m, \tag{66}$$

$$z_1^{(3)} = (\lambda_3 - \lambda_2)l_0 = 8.41\mu m, \tag{67}$$

$$z_2^{(3)} = z_1^{(2)} + z_1^{(3)} = 18.68\mu m, \tag{68}$$

The three state (ijk) superposition probability density at position of zero z_i^k is equal to the two state superposition (jk) probability:

$$|\Phi_{(ijk)}(z_n^{(k)}, t)|^2 = |\Phi_{(ij)}(z_n^{(k)}, t)|^2. \tag{69}$$

This means that the three state probability density exhibits harmonic time-dependent oscillation with a frequency ω_{ij} at height of zero z_n^k. Let us mention that the time dependence of $|\Phi_{ijk}(z,t)|^2$ in any position z, except for the mentioned zeros is not harmonic; it is given by a superposition of three cosine functions with different frequencies, analogous to Eq(54). This property allows us to extract the zeros positions using the probability density. One can see that the knowledge of zeros (Eq.(67-68)) is analogous to the knowledge of the transition frequencies. A measurement of one zero position allows us to extract the spatial scale l_0, a measurement of positions of two or more zeros allows us to constrain hypothetical nonlinear

additions to the gravitational potential. We show the probability density as a function of height z above the mirror (y-axis) and the time t (x-axis) in a superposition of first, second and third gravitational state in Fig.5.
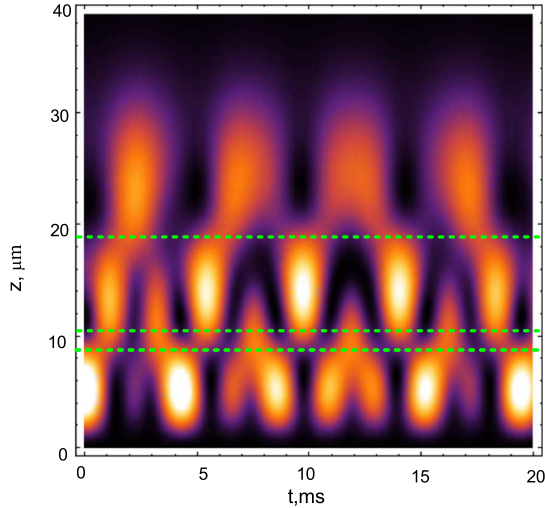

FIG. 5: Color. The probability density of $\bar{H}$ in a superposition of the first, second and third gravitational states, as a function of the height z above a mirror (vertical-axis) and the time t (horizontal-axis). Dark shade: low probability density. Light shade: high probability density. The dashed lines indicate the positions of the nodes in the wave-functions of the second and third state.

V. FEASIBILITY OF ANTIHYDROGEN GRAVITATIONAL STATES EXPERIMENT

In this section we study the feasibility of an experiment on the gravitationally bound quantum states of anti-hydrogen atoms. For such an estimation, we compare it with the already performed gravitational states experiments using ultra-cold neutrons (UCN) [17–19]. The UCN gravitational experiments can be used as a benchmark for such a comparison because 1) the neutron mass is nearly equal to the anti-hydrogen mass, 2) any modification to the $\bar{H}$ quantum state energies and wave-functions following from the precise shape of the Casimir-Polder potential are small compared to those of the quantum bouncer; UCN in the Earth's gravitational field above a perfect mirror is well described by the quantum bouncer model, 3) our estimation of $\bar{H}$ lifetimes in the quantum states of 0.1 s are compatible to or even longer than the time of UCN passage through the mirror-absorber installation, also 4)

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UCN velocities are comparable to velocities of ultra-cold $\bar{H}$ atoms produced in traps [13, 14]. We will discuss here mainly the statistical limitations arising from an estimate of the spectra from sources of $\bar{H}$ atoms that are projected in the near future.

In the simplest configuration, the experimental method for observation of the neutron gravitational states consisted in measuring the UCN flux through a slit between the horizontal mirror and the flat absorber (scatterer) placed above it at a variable height as a function of the slit height (the integral measuring method), or analyzing the spatial UCN density distribution behind the horizontal bottom mirror exit (the differential measuring method) using position-sensitive neutron detectors. The slit height can be changed and precisely measured. The absorber acts selectively on the gravitational states, namely the states with a spatial size $H_n = \lambda_n^0 l_0$ smaller than the absorber height H are weakly affected, while the states with $H_n > H$ are intensively absorbed [24, 28, 29]. A detailed description of the experimental method, the experimental setup, the results of various applications of this phenomenon could be found, for instance in [17–19, 30?].

Leaving aside numerous methodical difficulties in the experiments of this kind (as they have been already overcome in the neutron experiments) and a real challenge to get high phase-space densities of trapped anti-hydrogen atoms (they are aimed at anyway in the existing anti-hydrogen projects [10, 11]), let us compare relevant phase-space densities in the two problems, keeping in mind that it is the principle parameter, which defines population of quantum states in accordance with the Liouville theorem. If the phase-space densities of anti-hydrogen atoms would be equal to those of UCN, we would just propose to use an existing UCN gravitational spectrometer [31, 32] for anti-hydrogen experiments with minor modifications.

UCN is an extremely narrow initial fraction in a much broader, and hotter neutron velocity distribution. Maximum UCN fluxes available today for experiments in a flow-though mode are equal to 4 10^3 $UCN/cm^2/s$; such UCN populate uniformly the phase-space up to the so-called critical velocity of about 6 m/s (UCN with smaller velocity are totally reflected from surface under any incidence angle; thus they could be stored in closed traps and transported using UCN guides). If one uses pulsed mode with a duty cycle, say 10^{-3}, the average flux would drop to 4 UCN/cm^2/s. The pulse method provides more precise measurements, it is used in current experiments with the GRANIT spectrometer, and it will be used in gravitational interference measurements analogous to those performed with the

centrifugal quantum states of neutrons [33, 34]. Taking into account the phase-space volume available for UCN in the gravitationally bound quantum states in the GRANIT spectrometer [32], we estimate the total count rate of about 10^2 events/day when the relative accuracy for the gravitational mass is 10^{-3} (we note that the accuracy in the mentioned experiment is defined by a width of a quantum transition, and a few events might be sufficient to observe the corresponding resonance).

The average flux of $\bar{H}$ atoms projected by AEGIS collaboration [15, 35] is a few atoms per second; let's take it equal 3 $\bar{H}/s$ to have it defined. The cloud length is $\sim$ 8 mm, its radius is $\sim$ 1.5 mm, thus the cloud volume is $\sim 5 \times 10^{-2}$ cm^3. For comparison, estimate of the average UCN flux, which would be emitted from a small UCN source with a volume of $\sim 5 \times 10^{-2}$ cm^3, the maximum UCN density available of 30 UCN/cm^3, with a duty cycle of $\sim 10^{-3}$ gives 0.15 UCN/s. It is 20 times lower than the $\bar{H}$ flux estimated above. One should not forget, however, that the projected $\bar{H}$ temperature is $\sim$ 100 μK, i.e. 100 times larger than the effective UCN temperature. Thus we lose a factor of $\sqrt{100} = 10$ because of larger spread of $\bar{H}$ vertical velocities. No geometrical factors are taken into account here as well as no constraints following from final solid angles allowed, final sizes of $\bar{H}$ detectors, mirrors etc. However, their account would decrease our estimation by only a few times provided proper experiment design (note that equal acceleration of all anti-hydrogen atoms would not decrease their phase-space density). Thus we could provide statistical power of $\bar{H}$ experiment compatible to that with UCN. As the projected temperature of anti-hydrogen atoms in another proposal [36] is significantly lower ($\sim$ 1 μK that is just equal to the effective UCN temperature), the phase-space density of anti-hydrogen atoms could be even higher. Another significant advantage of a lower temperature consists in a more compact setup design. Note that a gravitational spectrometer analogous to [31, 32] selects just a very small fraction of UCN($\bar{H}$) available (those with extremely small vertical velocity components) thus the count rate of "useful" events is extremely low in both cases.

Thus, we conclude that measurements of the gravitationally bound quantum states of anti-hydrogen atoms look realistic if they would profit from methodical developments available in neutron experiments plus high phase-space densities of anti-hydrogen atoms aimed at in future. Based on extensive analysis of the mentioned neutron experiments, we could conclude that measurements of the gravitational mass of anti-hydrogen atoms with an accuracy of at least 10^{-3} is realistic, provided that projected high $\bar{H}$ phase-space density is

achieved.

VI. CONCLUSIONS

We argue on existence of long-living quasi-stationary states of $\bar{H}$ above a material surface in the gravitational field of Earth. A typical lifetime of such states above an ideally conducting plane surface is $\tau \simeq 0.1$ s. Quasi-stationary character of such states is due to the quantum reflection of ultra-cold (anti)atoms from the Casimir-Polder (anti)atom-surface potential. The relatively long life-time is due to the smallness of the ratio of the characteristic spatial antiatom-surface interaction scale l_{CP} and the spatial gravitational scale l_0. We show that the spectrum of $\bar{H}$ decaying gravitational levels is quasi-discrete even for the highly excited states as long as their quantum number $n \ll 30000$. We argue that low lying gravitational states provide an interesting tool for studying the gravitational properties of antimatter, in particular for testing the equivalence between the gravitational and inertial masses of $\bar{H}$. We show that, by counting the number of $\bar{H}$ annihilation events on the surface, both the transition frequencies between the gravitational energy levels, as well as the spatial density distribution of gravitational states superposition can be measured. An important observation in this context is that a modification of the above mentioned properties of gravitational states due to the interaction with a surface disappears in the first order of the small ratio l_{CP}/l_0. Finally, we show that actual measurements of quantum properties of $\bar{H}$ atoms, levitating above a material mirror in gravitational states are feasible, provided that the projected high phase-space density $\bar{H}$ is achieved.

VII. ACKNOWLEDGMENTS

VIII.   APPENDIX

Here we derive an expression for the scalar product of two complex energy gravitational states eigenfunctions:

$$\alpha_{ij} = \frac{1}{N_i N_j} \int_0^\infty \mathrm{Ai}^*(z/l_0 - \lambda_j)\, \mathrm{Ai}(z/l_0 - \lambda_i)dz \tag{70}$$

with

$$N_i^2 = \int_0^\infty \mathrm{Ai}^2(z/l_0 - \lambda_j)dz. \tag{71}$$

We start with the equations for the eigenfunction $\mathrm{Ai}(z/l_0 - \lambda_i)$ and the complex eigenvalues $\lambda_i = \lambda_i^0 + a_{CP}/l_0$:

$$-\mathrm{Ai}''(z/l_0 - \lambda_i) + z\,\mathrm{Ai}(z/l_0 - \lambda_i) = \lambda_i\,\mathrm{Ai}(z/l_0 - \lambda_i). \tag{72}$$

The equations for the complex conjugated eigenfunction and the eigenvalue are:

$$-\mathrm{Ai}^{*''}(z/l_0 - \lambda_j) + z\,\mathrm{Ai}^*(z/l_0 - \lambda_j) = \lambda_j^*\,\mathrm{Ai}^*(z/l_0 - \lambda_j). \tag{73}$$

We multiply both sides of equation Eq.(72) by $\mathrm{Ai}^*(z/l_0 - \lambda_j)$ and integrate them over z. Then we multiply both sides of Eq.(73) by $\mathrm{Ai}(z/l_0 - \lambda_i)$ and integrate them over z. After substraction of the results of these operations we get:

$$\mathrm{Ai}^{*'}(-\lambda_j)\,\mathrm{Ai}(-\lambda_i) - \mathrm{Ai}^*(-\lambda_j)\,\mathrm{Ai}'(-\lambda_i) = (\lambda_j^* - \lambda_i)\int_0^\infty \mathrm{Ai}^*(z/l_0 - \lambda_j)\,\mathrm{Ai}(z/l_0 - \lambda_i)dz. \tag{74}$$

To get the above result, we integrated by parts the integrals with second derivatives and took into account that Airy functions vanish at infinity. Now we take into account the equality $\mathrm{Ai}(-\lambda_i^0) = 0$ and smallness of the ratio a_{CP}/l_0, to get the following expressions, exact up to the second order in a_{CP}/l_0:

$$\mathrm{Ai}(-\lambda_i) = -\frac{a_{CP}}{l_0}\,\mathrm{Ai}'(-\lambda_i^0), \tag{75}$$

$$\mathrm{Ai}^*(-\lambda_j) = -\frac{a_{CP}^*}{l_0}\,\mathrm{Ai}^{*'}(-\lambda_j^0). \tag{76}$$

Up to the second order in a_{CP}/l_0 we get:

$$\mathrm{Ai}^{*'}(-\lambda_j)\,\mathrm{Ai}(-\lambda_i) - \mathrm{Ai}^*(-\lambda_j)\,\mathrm{Ai}'(-\lambda_i) = i\frac{b}{2l_0}\,\mathrm{Ai}^{*'}(-\lambda_j^0)\,\mathrm{Ai}'(-\lambda_i^0). \tag{77}$$

The above result should be combined with the known expression for the normalization coefficient:

$$N_i^2 = \mathrm{Ai}'^2(-\lambda_i) + \lambda_i \,\mathrm{Ai}^2(-\lambda_i), \tag{78}$$

which, up to the second order in a_{CP}/l_0 turns to be:

$$N_i = \mathrm{Ai}'(-\lambda_i^0)(1 + \lambda_i^0 \frac{a_{CP}^2}{2l_0^2}). \tag{79}$$

Keeping first order terms, we finally get:

$$\alpha_{i \neq j} = i \frac{b/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)}. \tag{80}$$

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