Gravitational quantum states of Antihydrogen

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We present a theoretical study of the motion of the antihydrogen atom ($\bar{\mathrm{H}}$) in the gravitational field of Earth above a material surface. We predict that the $\bar{\mathrm{H}}$ atom, falling in the gravitational field of Earth above a material surface, would settle into long-lived quantum states. We point out a method of measuring the difference in the energy of $\bar{\mathrm{H}}$ in such states. The method allows for spectroscopy of gravitational levels based on atom-interferometric principles. We analyze the general feasibility of performing experiments of this kind. We point out that such experiments provide a method of measuring the gravitational force (Mg) acting on $\bar{\mathrm{H}}$ and that they might be of interest in the context of testing the weak equivalence principle for antimatter.

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I. INTRODUCTION

Galileo, Newton, and Einstein recognized that all bodies, regardless of their mass and composition, fall toward the Earth with an equal gravitational acceleration. Is that conclusion valid for antimatter? This has never been tested.

In the context of the general relativity theory, the universality of free fall is often referred to as the weak equivalence principle (WEP). Violations of the WEP could occur in ordinary matter-matter interactions—e.g., as a result of the difference between gravitational coupling to the rest mass and that to the binding energy. The WEP is being tested with increasing sensitivity for macroscopic bodies. The best test so far confirms the WEP to an accuracy of 2×10^{-13} (using a rotating torsion balance [1]). Ongoing projects aim for an accuracy of 1 part in 10^{16} (laser tracking of a pair of test bodies in a freely falling rocket [2]), or even of 1 part in 10^{18} (in an Earth-orbiting satellite [3]). However, in view of the difficulties in unifying quantum mechanics with the theory of gravity, it is of great interest to investigate the gravitational properties of *quantum mechanical objects*, such as elementary particles or atoms. Such experiments have already been performed—e.g., using interferometric methods to measure the gravitational acceleration of neutrons [4,5] and atoms [6–9]. However, the experiments with *anti*atoms (see [10,11] and references therein) are even more interesting in view of testing the WEP, because the theories striving to unify gravity and quantum mechanics (such as supersymmetric string theories) tend to suggest violation of the gravitational equivalence of particles and antiparticles [12]. Experiments testing gravitational properties of antiatoms are on the agenda of all experimental groups working with antihydrogen (see, e.g., ATHENA-ALPHA [13], ATRAP [14], and AEGIS [15]). One of the challenging aspects of experiments of this kind is to control the initial parameters of antiatoms, such as their temperature and position, with sufficient accuracy [16].

In the present paper we investigate the possibility of exploring gravitational properties of antiatoms in the ultimate quantum limit. We study antihydrogen atoms levitating in the lowest gravitational states above a material surface. The existence of such gravitational states for *neutrons* was proven experimentally [17–19]. The existence of analogous states for antiatoms seems, at a first glance, impossible because of annihilation of antiatoms in the material walls. However, we have shown that ultracold antihydrogen atoms are efficiently reflected from a material surface [20,21] because of so-called quantum reflection from the Casimir-Polder atom-surface interaction potential. We have shown that antihydrogen atoms, confined by quantum reflection via Casimir forces from below and by gravitational force from above, would form metastable gravitational quantum states. They would bounce on a surface for a finite lifetime (of the order of 0.1 s) [20]. This simple system can be considered a microscopic laboratory for testing the gravitational interaction under extremely well specified (in fact, quantized!) conditions.

The annihilation of ultraslow antiatoms in a wall occurs with a small but finite (a few percent) probability. It provides a clear and easy-to-detect signal, which might be used to measure continuously the antiatom density in the gravitational states as a function of time. If antiatoms are settled in a superposition of gravitational states, the antiatom density evolves with beatings, determined by the *energy difference* between the gravitational levels. The transition frequencies between the gravitational levels are related to the strength of the gravitational force Mg acting on antiatoms; here M is the gravitational mass of $\bar{\mathrm{H}}$, and g is the Earth's local gravitational field strength. We also show that a measurement of *differences* between the energy levels would allow us to disentangle Mg in a way independent of the effects of the antiatom-surface interaction.

The plan of the paper is as follows. In Sec. II we study the main properties of the quasistationary gravitational states, in Sec. III we present the time evolution of the superposition of $\bar{\mathrm{H}}$ gravitational states, and in Sec. IV we discuss the concept of a quantum ballistic experiment, namely, the spatial-temporal evolution of the superposition of $\bar{\mathrm{H}}$ gravitational states. In Sec. V we analyze the feasibility of measuring $\bar{\mathrm{H}}$ atom properties in gravitational quantum states. In the Appendix we derive useful analytical expressions for the scalar product of quasistationary gravitational states.

II. GRAVITATIONAL STATES OF $\bar{\mathrm{H}}$

In this section we discuss the properties of the gravitational states of $\bar{\mathrm{H}}$ above a material surface.

We consider an $\bar{\mathrm{H}}$ atom bouncing above a material surface in the gravitational field of Earth. $\bar{\mathrm{H}}$ is confined because of quantum scattering from the Casimir-Polder potential below and the gravitational field above. The Schrödinger equation for the wave function $\Psi(z)$ of $\bar{\mathrm{H}}$ in such a superposition of atom-surface and gravitational potentials is

$$\left[-\frac{\hbar^2\partial^2}{2m\partial z^2}+V(z)+Mgz-E\right]\Psi(z)=0. \quad (1)$$

Here z is the distance between the surface and the $\bar{\mathrm{H}}$ atom, and $V(z)$ is the atom-surface interaction potential with a long-range asymptotic form $V(z)\sim -C_4/z^4$. We distinguish between the gravitational mass, which we refer to as M, and the inertial mass, denoted by m, hereafter. The wave function $\Psi(z)$ satisfies the full absorption boundary condition at the surface ($z=0$) [21], which stands for the annihilation of antiatoms in the material wall.

The characteristic length and energy scales are

$$l_0=\sqrt[3]{\frac{\hbar^2}{2mMg}}, \quad (2)$$

$$l_{\mathrm{CP}}=\sqrt{2mC_4}, \quad (3)$$

$$\varepsilon_0=\sqrt[3]{\frac{\hbar^2M^2g^2}{2m}}, \quad (4)$$

$$\varepsilon_{\mathrm{CP}}=\frac{\hbar^2}{4m^2C_4}. \quad (5)$$

Here $l_0=5.871$ μm is the characteristic gravitational length scale, $l_{\mathrm{CP}}=0.027$ μm is the characteristic Casimir-Polder interaction length scale, $\varepsilon_0=2.211\times10^{-14}$ a.u. is the characteristic gravitational energy scale, and $\varepsilon_{\mathrm{CP}}=1.007\times10^{-9}$ a.u. is the Casimir-Polder energy scale. As one can see, the gravitational length scale is much larger than the Casimir-Polder length scale ($l_0\gg l_{\mathrm{CP}}$), while the gravitational energy scale is much smaller than the Casimir-Polder energy scale ($\varepsilon_0\ll\varepsilon_{\mathrm{CP}}$). It is also useful to introduce the gravitational time scale τ_0:

$$\tau_0=\hbar/\varepsilon_0\simeq 0.001\ \mathrm{s}. \quad (6)$$

For large atom-surface separation distances $z\gg l_{\mathrm{CP}}$ the solution of Eq. (1) has the form

$$\Psi(z)\sim \mathrm{Ai}\left(\frac{z}{l_0}-\frac{E}{\varepsilon_0}\right)+K(E)\,\mathrm{Bi}\left(\frac{z}{l_0}-\frac{E}{\varepsilon_0}\right), \quad (7)$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions [22]. The requirement of square integrability of the wave function $\Psi(z\to\infty)\to 0$ results in the following equation for the energy levels of the gravitational states in the presence of the Casimir-Polder interaction:

$$K(E_n)=0. \quad (8)$$

The hierarchy of the Casimir-Polder and gravitational scales ($l_{\mathrm{CP}}\ll l_0$) suggests that the quantum reflection from the Casimir-Polder potential can be accounted for by modifying the boundary condition for the quantum bouncer (a particle bouncing in the gravitational field above a surface; the interaction of the latter with a particle is modeled by an infinite reflecting wall). The wave-function of the quantum bouncer satisfies the following equation system:

$$\left(-\frac{\hbar^2\partial^2}{2m\partial z^2}+Mgz-E_n\right)\Phi_n(z)=0, \qquad \Phi_n(z=0)=0. \quad (9)$$

The energy levels of the quantum bouncer are known to be [17]

$$E_n^0=\varepsilon_0\lambda_n^0, \quad (10)$$

$$\mathrm{Ai}\left(-\lambda_n^0\right)=0. \quad (11)$$

TABLE I. The eigenvalues, gravitational energies, and classical turning points of a quantum bouncer with the mass of (anti)hydrogen in the Earth's gravitational field.

n	λ_n^0	E_n^0 (peV)	z_n^0 (μm)
1	2.338	1.407	13.726
2	4.088	2.461	24.001
3	5.521	3.324	32.414
4	6.787	4.086	39.846
5	7.944	4.782	46.639
6	9.023	5.431	52.974
7	10.040	6.044	58.945

Table I summarizes the eigenvalues and classical turning points $z_n^0=E_n^0/(Mg)$ for the first seven gravitational states of a quantum bouncer (with the mass of antihydrogen).

For the distances $l_{\mathrm{CP}}\ll z\ll l_0$ one could neglect the gravitational potential in Eq. (1). In this approximation, the solution of Eq. (1) has the following asymptotic form:

$$\Psi(z)\sim\sin[kz+\delta(E)]. \quad (12)$$

Here k is the wave vector $k=\sqrt{2mE}$, and $\delta(E)$ is the phase shift of $\bar{\mathrm{H}}$ reflected from the Casimir-Polder potential *in the absence of the gravitational field* [21]. Matching the asymptotics in Eq. (12) and Eq. (7), we get the relation between the phase shift $\delta(E)$ and the K function introduced in Eq. (7):

$$K(E)=-\frac{\tan[\delta(E)]\,\mathrm{Ai}'(-E/\varepsilon_0)-kl_0\,\mathrm{Ai}(-E/\varepsilon_0)}{\tan[\delta(E)]\,\mathrm{Bi}'(-E/\varepsilon_0)-kl_0\,\mathrm{Bi}(-E/\varepsilon_0)}. \quad (13)$$

In deriving the above expression we took into account that the relation between $K(E)$ and $\delta(E)$ should not depend on the matching point z_m and thus can be formally attributed to $z_m=0$. Substitution of Eq. (13) into Eq. (8) gives an equation for the *distorted* gravitational levels:

$$\frac{\tan[\delta(E_n)]}{kl_0}=\frac{\mathrm{Ai}(-E_n/\varepsilon_0)}{\mathrm{Ai}'(-E_n/\varepsilon_0)}. \quad (14)$$

This equation is equivalent to the following boundary condition:

$$\frac{\Phi(0)}{\Phi'(0)}=\frac{\tan[\delta(E_n)]}{k}. \quad (15)$$

Thus an $\bar{\mathrm{H}}$ atom, confined by the gravitational field of Earth and the quantum reflection from the Casimir-Polder potential, can be described by the following equation system:

$$\left(-\frac{\hbar^2\partial^2}{2m\partial z^2}+Mgz-E_n\right)\Phi_n(z)=0,\qquad \frac{\Phi(0)}{\Phi'(0)}=\frac{\tan[\delta(E_n)]}{k}. \tag{16}$$

For the lowest gravitational states the condition $kl_{\rm CP}\ll 1$ is valid. Thus the scattering-length approximation for the phase shift $\delta(E)\approx -ka_{\rm CP}$ is well justified. The *complex-valued* quantity [21]

$$a_{\rm CP}=-(0.10+i1.05)l_{\rm CP}, \tag{17}$$

$$a_{\rm CP}=-0.0027-i0.0287\ \mu{\rm m} \tag{18}$$

is the scattering length for the Casimir-Polder potential provided full absorbtion in the material wall. Thus the equation for the lowest eigenvalues (14) takes a form

$$\frac{a_{\rm CP}}{l_0}=-\frac{{\rm Ai}(-E_n/\varepsilon_0)}{{\rm Ai}'(-E_n/\varepsilon_0)}. \tag{19}$$

The above equation is equivalent to the following boundary condition for the wave function $\Phi(z)$ of a particle in the gravitational potential Eq. (1):

$$\Phi(z\to 0)\to z-a_{\rm CP}. \tag{20}$$

Because of the imaginary part of the scattering length $a_{\rm CP}$, the gravitational states of $\bar{\rm H}$ above a material surface are *quasistationary decaying* states. For low quantum numbers n, it is easy to relate the lowest quasistationary energy levels E_n to the unperturbed gravitational energy levels E_n^0 of a quantum bouncer. Indeed, the variable substitution $z=\tilde{z}+a_{\rm CP}$ transforms Eqs. (9) and (20) into the equation system for the quantum bouncer:

$$\left[-\frac{\hbar^2\partial^2}{2m\partial\tilde{z}^2}+Mg\tilde{z}-(E_n-Mga_{\rm CP})\right]\Phi_n(\tilde{z})=0, \tag{21}$$

$$\Phi_n(\tilde{z}\to 0)\to 0. \tag{22}$$

The eigenvalues E_n and eigenfunctions Φ_n are

$$E_n=E_n^0+Mga_{\rm CP}, \tag{23}$$

$$\Phi_n(z)=\frac{1}{N_i}{\rm Ai}\left((z-a_{\rm CP})/l_0-\lambda_n^0\right), \tag{24}$$

where N_i is the normalization coefficient [see Eqs. (A9) and (A10) in the Appendix]. In the following, we will use the dimensionless eigenvalues $\lambda_n=E_n/\varepsilon_0$:

$$\lambda_n=\lambda_n^0+a_{\rm CP}/l_0. \tag{25}$$

An important message from the above expression is that the complex shift $Mga_{\rm CP}$ (due to the account of quantum reflection on the Casimir-Polder potential) is *the same* for all low-lying quasistationary gravitational levels. This means that the transition frequencies between the gravitational states are not affected by the Casimir-Polder interaction, provided the latter can be described by the complex scattering length $a_{\rm CP}$. The scattering-length approximation is valid in the limit $k_na_{\rm CP}\to 0$, where $k_n=\sqrt{2mE_n}$ (let us note that for the first gravitational state $|k_1a_{\rm CP}|=0.0071$). However, accounting for the higher order k-dependent terms in Eq. (14) would result in a state-dependent shift of the gravitational states due to the Casimir-Polder interaction. We use a known low-energy expansion of the s-wave phase shift $\delta(E)$ in a homogeneous $1/z^4$ potential [23], in which we keep the two leading k-dependent terms:

$$a_{\rm CP}(k)\cot[\delta(k)]\simeq -1+\frac{\pi}{3}\frac{l_{\rm CP}}{a_{\rm CP}}(l_{\rm CP}k)+\frac{4}{3}(l_{\rm CP}k)^2\ln\left(\frac{l_{\rm CP}k}{4}\right)+\ldots. \tag{26}$$

We introduce a k-dependent modified "scattering length" $\tilde{a}_{\rm CP}(k)\equiv-\delta(k)/k$ and get the following expression for $\tilde{a}_{\rm CP}(k)$:

$$\tilde{a}_{\rm CP}(k)\simeq a_{\rm CP}+\frac{\pi}{3}l_{\rm CP}(l_{\rm CP}k)+\frac{4}{3}a_{\rm CP}(l_{\rm CP}k)^2\ln\left(\frac{l_{\rm CP}k}{4}\right). \tag{27}$$

The leading k-dependent term in the above expression, $\frac{\pi}{3}l_{\rm CP}(l_{\rm CP}k)$, is real and independent of properties of the inner part of the Casimir-Polder interaction. It is determined by the asymptotic form of the potential, and thus it depends on the Casimir-Polder length scale $l_{\rm CP}$ only. Thus the modified equation for the gravitational-state energies is

$$E_n=E_n^0+Mg\tilde{a}_{\rm CP}(E_n). \tag{28}$$

Taking into account the smallness of the k-dependent terms (for the lowest gravitational states) in expression (27), we get

$$E_n\simeq E_n^0+Mg\tilde{a}_{\rm CP}(k_n^0)=\varepsilon\left[\lambda_n^0+a_{\rm CP}/l_0+\frac{\pi l_{\rm CP}}{3l_0}(l_{\rm CP}k_n^0)+\frac{4a_{\rm CP}}{3l_0}(l_{\rm CP}k_n^0)^2\ln\left(\frac{l_{\rm CP}k_n^0}{4}\right)\right]. \tag{29}$$

Here $k_n^0=\sqrt{2mE_n^0}$.

Accounting for k-dependent terms in Eq. (27) modifies the transition frequencies between the gravitational states in a way that is dependent on the Casimir-Polder interaction. However, such modification is very weak. Indeed, taking into account that $l_{\rm CP}k_n^0\sim l_{\rm CP}/l_0$ for the lowest gravitational states, the leading k-dependent corrections to the gravitational energy are of the second order in the small parameter $l_{\rm CP}/l_0$. The transition frequency between the first and second gravitational states equals $\omega_{12}=\omega_{12}^0+\Delta_{12}$, where $\omega_{12}^0=(E_2^0-E_1^0)/(2\pi\hbar)=254.54$ Hz, and $\Delta_{12}=Mg[\tilde{a}_{\rm CP}(k_2^0)-\tilde{a}_{\rm CP}(k_1^0)]=0.0017$ Hz.

An account of the first two terms in Eq. (27) provides an equal decay width for the lowest gravitational states. This width is determined by the probability of antihydrogen penetrating to the surface and annihilating:

$$\Gamma_n=\varepsilon\frac{b}{2l_0}. \tag{30}$$

Here we use a standard notation $b = 4\,\mathrm{Im}\,a_{CP}$. According to our previous calculations [20]

$$b = 0.115\ \mu\text{m}. \tag{31}$$

The widths of the gravitational states (30) are proportional to the ratio ε/l_0. Using Eqs. (2) and (4) we find that this ratio is equal to the gravitational force $\varepsilon/l_0 = Mg$ so that

$$\Gamma_n = \frac{b}{2}Mg. \tag{32}$$

The corresponding lifetime (calculated for an ideal conducting surface) is

$$\tau = \frac{2\hbar}{Mgb} \simeq 0.1\ \text{s}. \tag{33}$$

We note factorization of the gravitational effect (appearing in this formula via the factor Mg) and the quantum reflection effect, manifesting through the constant b. Such a factorization is a consequence of the smallness of the ratio of the characteristic scales $b/(2l_0) \simeq 0.01$.

Comparing the life time of $\bar{\mathrm{H}}$ in one of the lowest gravitational states to the classical period $T = 2\sqrt{\frac{2l_0\lambda_1}{g}} \simeq 0.0033$ s of $\bar{\mathrm{H}}$ bouncing with the energy of the ground state, we see that $\bar{\mathrm{H}}$ bounces an average of about 30 times before annihilating. This shows that the lowest gravitational states are well-resolved quasistationary states.

It is interesting to estimate the upper limit of the quantum number N, below which the gravitational states is still resolved; i.e,

$$\frac{\tau_N}{T_N} = \frac{2\pi\hbar}{\Gamma(N)\frac{dE(n)}{dn}} > 1. \tag{34}$$

Here τ_N is the lifetime of the Nth gravitational state, and T_N is a classical period corresponding to the Nth state via $T_N = 2\hbar\pi/(dE(n)/dn)_{|n=N}$. For such an estimation we transform Eq. (14) using the asymptotic form of the Airy function for a large negative argument and get

$$\lambda_n = \left\{\frac{3}{2}\left[\pi\left(n - \frac{1}{4}\right) - \delta(E_n)\right]\right\}^{2/3}. \tag{35}$$

The accuracy of this equation increases with increasing n; it gives the energy value within a few percent even for $n = 1$. In the energy domain of interest $|\delta(E)| \ll \pi(n - \frac{1}{4})$ [21], so

$$\lambda_n \simeq \lambda_n^0 - \frac{\delta(E_n^0)}{\sqrt{\lambda_n^0}}. \tag{36}$$

Here we use the semiclassical approximation for λ_n^0 [24]:

$$\lambda_n^0 \simeq \left[\frac{3}{2}\pi\left(n - \frac{1}{4}\right)\right]^{2/3}. \tag{37}$$

One can verify that in the case of small n Eq. (37) reduces to Eq. (25). The substitution of Eq. (36) into Eq. (34) results in

$$\frac{\tau_n}{T_n} \simeq \frac{1}{4\,\mathrm{Im}\,\delta(E_n^0)}. \tag{38}$$

The ratio $\tau(n)/T(n)$ expresses the number of classical bounces of $\bar{\mathrm{H}}$ during the lifetime of its nth state. This dependence is shown in Fig. 1. Using numerical values $\delta(E)$, calculated in [21], we find that inequality (34) holds for

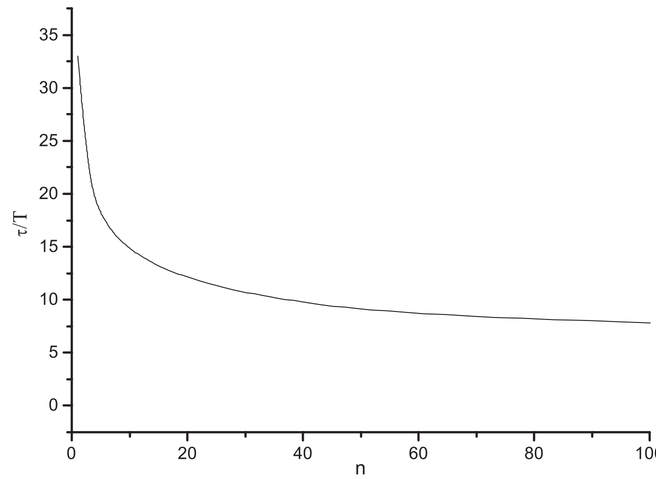

FIG. 1. The number of $\bar{\mathrm{H}}$ bounces during the lifetime of the nth gravitational state.

$$n < N = 30\,000. \tag{39}$$

The corresponding energy $E_N = 6 \times 10^{-11}$ a.u., and the characteristic size of such states is as large as $H_N = 1.6$ cm. This means that the concept of the quasibound gravitational states is justified not only for the lowest states; it also might be applied for highly excited states.

The quasistationary character of the gravitational states of the antiatom above a material surface manifests itself in a nonzero current through the bottom surface ($z = 0$). Indeed, the expression for the current is

$$j(z,t) = \frac{i\hbar}{2M}\left[\Phi(z,t)\frac{d\Phi^*(z,t)}{dz} - \Phi^*(z,t)\frac{d\Phi(z,t)}{dz}\right], \tag{40}$$

which taken at $z = 0$ for a given gravitational state (24) turns out to be equal to

$$j(0,t) = \varepsilon \exp\left(-\frac{\Gamma}{\hbar}t\right) \times \frac{\mathrm{Ai}^*(-\lambda_n)\,\mathrm{Ai}'(-\lambda_n) - \mathrm{Ai}(-\lambda_n)\,\mathrm{Ai}'^*(-\lambda_n)}{N_i N_i^*}. \tag{41}$$

Here λ_n is given by Eq. (25), and N_i is the normalization factor. We take into account the smallness of the ratio a_{CP}/l_0 and Eq. (11), and get

$$\mathrm{Ai}(-\lambda_n) \approx -\frac{a_{\mathrm{CP}}}{l_0}\,\mathrm{Ai}'\left(-\lambda_n^0\right), \tag{42}$$

which is exact up to the second order in the ratio a_{CP}/l_0. Now taking into account an explicit form of the normalization coefficients [Eq. (A10) in the Appendix] $N_i = \mathrm{Ai}'(-\lambda_n^0)$, we get finally

$$j(0,t) = -\varepsilon\frac{b}{2\hbar l_0}\exp\left(-\frac{\Gamma}{\hbar}t\right) = -\frac{\Gamma}{\hbar}\exp\left(-\frac{\Gamma}{\hbar}t\right). \tag{43}$$

This result is in full agreement with Eq. (30) as far as

$$\frac{d}{dt}\int_0^\infty |\Phi(z,t)|^2 dz = j(0,t) = -\frac{\Gamma}{\hbar}\exp\left(-\frac{\Gamma}{\hbar}t\right). \tag{44}$$

III. BOUNCING ANTIHYDROGEN

In this section, we are interested in the evolution of an initially prepared arbitrary superposition of several lowest gravitational states of $\bar{\mathrm{H}}$. In the following, we will limit our treatment to the scattering-length approximation in the description of the Casimir-Polder interaction, and we will neglect all but the first term of expression (27), so that $\tilde{a}_{\mathrm{CP}}(k) \approx a_{\mathrm{CP}}$. The corresponding $\bar{\mathrm{H}}$ wave function is

$$\Phi(z,t) = \sum_{i=1}^{n} \frac{C_i}{N_i} \mathrm{Ai}(z/l_0 - \lambda_i) \exp\left(-i\lambda_i \frac{t}{\tau_0}\right). \quad (45)$$

Here τ_0 is the characteristic time scale of the $\bar{\mathrm{H}}$ bouncer, C_i are expansion coefficients, and $N_i = \mathrm{Ai}'(-\lambda_i)$ are the normalization factors of the gravitational states (see the Appendix).

We are interested in the evolution of the number of antihydrogen atoms as a function of time:

$$F(t) = \int_0^\infty |\Phi(z,t)|^2 dz = \sum_{i,j=1}^{n} \int_0^\infty \frac{C_j^* C_i}{N_j^* N_i} \mathrm{Ai}^*(z/l_0 - \lambda_j) \times \mathrm{Ai}(z/l_0 - \lambda_i) \exp[-i\varepsilon(\lambda_i - \lambda_j^*)t]\, dz. \quad (46)$$

First, let us note that this expression for the total number of particles is no longer constant, because of the decay of the quasistationary gravitational states. Second, the quasistationary gravitational states corresponding to different energies are nonorthogonal:

$$\frac{1}{N_i N_j} \int_0^\infty \mathrm{Ai}^*(z/l_0 - \lambda_j)\, \mathrm{Ai}(z/l_0 - \lambda_i)\, dz \equiv \alpha_{ij} \neq \delta_{ij}. \quad (47)$$

In the Appendix we will derive the following expression for the cross terms α_{ij}, exact up to the second order of the small ratio a_{CP}/l_0:

$$\alpha_{i\neq j} = i \frac{b/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)}. \quad (48)$$

As one can see, such cross terms vanish if there is no decay, i.e., if $b = 4\,\mathrm{Im}\, a_{\mathrm{CP}} \to 0$.

Now we can calculate an expression for the number of antihydrogen atoms as a function of time (46):

$$F(t) = \exp\left(-\frac{\Gamma}{\hbar}t\right)\left[\sum_i |C_i|^2 + 2\,\mathrm{Re} \sum_{i>j}^{n} \sum_{j}^{n} C_j^* C_i \times \frac{ib/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)} \exp\left(-i\left(\lambda_i^0 - \lambda_j^0\right)\frac{t}{\tau_0}\right)\right]. \quad (49)$$

From Eqs. (49) and (30) we get the following expression for the disappearance (annihilation) rate $-\frac{dF(t)}{dt}$, keeping the terms up to the second order in the ratio a_{CP}/l_0:

$$\frac{dF(t)}{dt} = -\frac{\Gamma}{\hbar} \exp\left(-\frac{\Gamma}{\hbar}t\right)\left[\sum_i |C_i|^2 + 2\,\mathrm{Re} \sum_{i>j}^{n} \sum_{j}^{n} C_j^* C_i \times \exp\left(-i\left(\lambda_i^0 - \lambda_j^0\right)\frac{t}{\tau_0}\right)\right]. \quad (50)$$

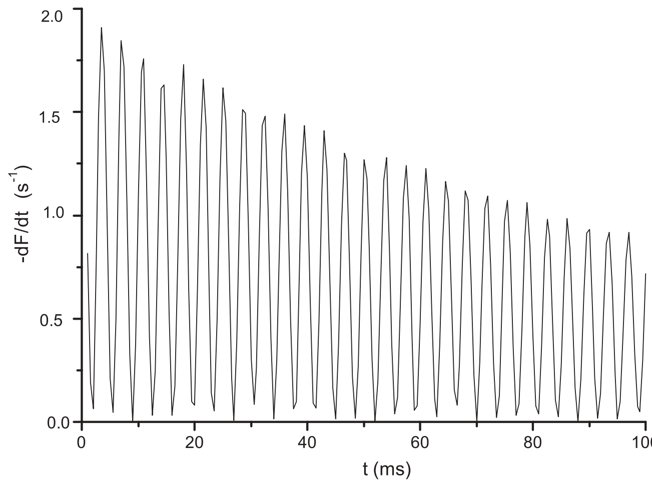

FIG. 2. Evolution of the annihilation rate of $\bar{\mathrm{H}}$ atoms in a superposition of the first and second gravitational states.

For a superposition of the two gravitational states with the equal coefficients C_1, C_2 (say, $C_1 = C_2 = 1$), the above expression attains a simple form:

$$\frac{dF_{12}(t)}{dt} = -\frac{\Gamma}{\hbar} \exp\left(-\frac{\Gamma}{\hbar}t\right)[1 + \cos(\omega_{12}t)]. \quad (51)$$

Here $\omega_{12} = (\lambda_2^0 - \lambda_1^0)/\tau_0$. The same result could be obtained by calculating the flux $j(0,t)$ Eq. (40) for a superposition of states (45).

One can see that the disappearance rate decays as a function of time according to the exponential law with the width Γ (the same for the lowest states); also, it oscillates with the transition frequency between the first and second gravitational states (equal to 254.54 Hz). We plot in Fig. 2 the time evolution of the disappearance rate for $\bar{\mathrm{H}}$ in a superposition of two lowest states. Curiously, the oscillation of the disappearance rate is the direct consequence of the decaying character of gravitational states. Indeed, such an oscillation is observable because of the nonvanishing contribution of the interference term in the expression for the total probability of finding antihydrogen atoms, given by Eq. (49). As one can see from Eq. (48), this contribution is proportional to the imaginary part of the scattering length; it would vanish in the case where there was no decay of gravitational states because of annihilation in the material wall (entering through the parameter b/l_0).

We observe that the oscillation frequency of the disappearance rate $F(t)$ corresponds to the energy difference between the unperturbed gravitational levels. Expression (50) does not include the shift of gravitational-state energies $\mathrm{Re}\, a_{\mathrm{CP}}/l_0$ since it is equal for all gravitational states, thus it is cancelled out in the energy difference. Accounting for higher order k-dependent terms in (27) would result in a small [second order of the ratio (a_{CP}/l_0)] correction to the transition frequency. A measurement of the oscillation frequency ω_{12} given by Eq. (51) would allow us to extract the following combination of the gravitational and inertial masses from Eq. (4):

$$\frac{M^2}{m} = \frac{2\hbar\omega_{12}^3}{g^2\left(\lambda_2^0 - \lambda_1^0\right)^3}. \quad (52)$$

Under the additional assumption of the equality of the known inertial mass of the *hydrogen* atom m_H and that of antihydrogen, imposed by *CPT*, we get

$$M = \sqrt{\frac{2m_H\hbar\omega_{12}^3}{g^2(\lambda_2^0-\lambda_1^0)^3}}. \tag{53}$$

The evolution of the superposition of *three* gravitational states provides information not only about the characteristic energy scale ε_0 but also about the level spacing as a function of quantum number n, characterized by the value $d^2E(n)/dn^2$. Such a study might be interesting for testing additional (to Newtonian gravitation) interactions (see [25,26] and references there in) between $\bar{\mathrm{H}}$ and a material surface within the spatial scale on the order of micrometers. Such interactions would manifest as nonlinear additions to the gravitational potential, which would modify the spectrum character. In the case of the superposition of three states, the disappearance rate (50) has the form

$$\frac{dF_{123}(t)}{dt} = -\frac{2}{3}\frac{\Gamma}{\hbar}\exp\left(-\frac{\Gamma}{\hbar}t\right)\left(\frac{3}{2}+\cos(\omega_{12}t)+\cos(\omega_{23}t) + \cos[(\omega_{12}+\omega_{23})t]\right). \tag{54}$$

Here $\omega_{ij} = (\lambda_j^0 - \lambda_i^0)/\tau_0$. One can verify that the period of coherence of $\cos(\omega_{12}t)$ and $\cos(\omega_{23}t)$ terms is

$$T_r = \frac{2\pi}{\omega_{12}-\omega_{23}} \simeq 0.02 \text{ s}. \tag{55}$$

A semiclassical expression for T_r is

$$T_r \approx \frac{2\pi}{|d^2E/dn^2|}. \tag{56}$$

One can see that the period T_r is a quantum limit analog of a half revival period $T_{rev} = 4\pi/|d^2E/dn^2|$ (T_{rev} characterizes the time period after which the evolution of the wave packet returns to semiclassical behavior; see [27] and references therein for details). In Fig. 3 we plot the annihilation events as a function of time (54) for a superposition of the three lowest gravitational states. The period T_r is clearly seen as a period of modulation of a rapidly oscillating function. The ratio

$$T_r/\tau_0 = \frac{2\pi}{\lambda_3 - 2\lambda_2 + \lambda_1} \tag{57}$$

is sensitive to any nonlinear addition to the gravitational potential. Indeed, while linear corrections to the gravitational potential can only change ε_0, nonlinear additions change the derivative of levels density $|d^2E/dn^2|$.

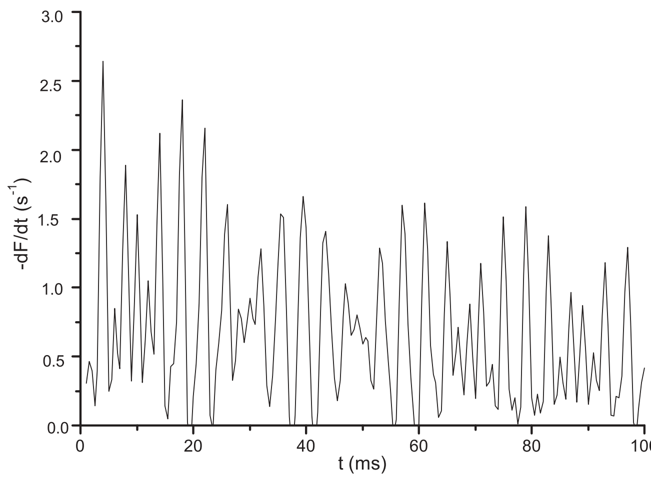

FIG. 3. Evolution of the annihilation rate of $\bar{\mathrm{H}}$ atoms in a superposition of the first, second, and third gravitational states.

IV. QUANTUM BALLISTIC EXPERIMENT

Two independent experiments are needed in order to determine the gravitational mass M and the inertial mass m of antihydrogen. In the previous section, we showed that a combination of gravitational and inertial masses M and m, given by Eq. (52), can be extracted from the frequency measurements, Eqs. (51) and/or (54). Independent information could be obtained from measurement of the spatial density distribution of $\bar{\mathrm{H}}$ in a superposition of the gravitation states, for instance, in the flow-through experiment (a kind of a beam-scattering experiment), in which $\bar{\mathrm{H}}$ atoms with a wide horizontal velocity distribution move along the mirror surface. The time of flight along the mirror should be measured simultaneously with the spatial density distribution in a position-sensitive detector, placed at the mirror exit. Such a detector would be able to measure the density distribution along the vertical axis at a given time instant. The horizontal component of the $\bar{\mathrm{H}}$ motion could be treated classically. Because of a broad distribution of the horizontal velocities in the beam, atoms would be detected within a wide range of time intervals between their entrance into the mirror and their detection at the exit. In such an approach, we could study the time evolution of $\bar{\mathrm{H}}$ probability density at a given position z:

$$|\Phi_{(12)}(z,t)|^2 = \exp\left(-\frac{\Gamma}{\hbar}t\right)\left[\left|\Phi_{(12)}^{av}(z)\right|^2 + 2\,\mathrm{Re}\,\frac{\mathrm{Ai}(z/l_0-\lambda_1)\,\mathrm{Ai}(z/l_0-\lambda_2)}{\mathrm{Ai}'(-\lambda_1)\,\mathrm{Ai}'(-\lambda_2)}\exp(-i\omega_{12}t)\right], \tag{58}$$

$$\left|\Phi_{(12)}^{av}(z)\right|^2 = \left|\frac{\mathrm{Ai}(z/l_0-\lambda_1)}{\mathrm{Ai}'(-\lambda_1)}\right|^2 + \left|\frac{\mathrm{Ai}(z/l_0-\lambda_2)}{\mathrm{Ai}'(-\lambda_2)}\right|^2. \tag{59}$$

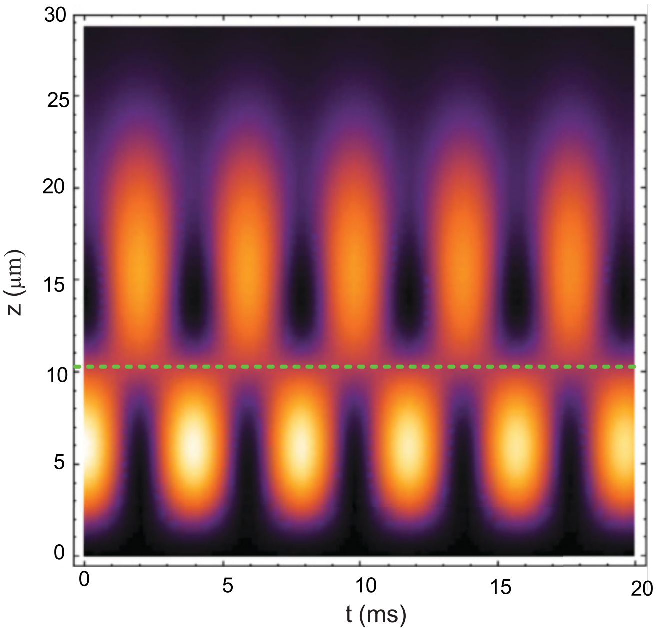

FIG. 4. (Color) The probability density of $\bar{\mathrm{H}}$ in a superposition of the first and second gravitational states, as a function of the height z above the mirror (vertical axis) and the time t (horizontal axis). Dark shade, low probability density; light shade, high probability density. The dashed line indicates the position of the node in the wave function of the second state.

The transition $\omega_{12} = 254.54$ Hz could be extracted from the time evolution of the probability density at a given z. The length scale l_0 could be extracted from the position of the zero $z_1^{(2)}$ of the wave function. Here the superscript stands for the state number and the subscript corresponds to the node number for a given state. Thus such a position is determined by the condition $\mathrm{Ai}(z_1^{(2)}/l_0 - \lambda_2) = 0$; it is equal to the following expression:

$$z_1^{(2)} = (\lambda_2 - \lambda_1) l_0 = 10.27\ \mu\mathrm{m}. \tag{60}$$

The probability density in Eq. (58) of a two-state superposition at $z = z_1^{(2)}$ behaves like the probability density of the ground state alone:

$$\left|\Phi_{12}\left(z_1^{(2)},t\right)\right|^2 = \exp\left(-\frac{\Gamma}{\hbar}t\right)\left|\frac{\mathrm{Ai}\left(z_1^{(2)}/l_0 - \lambda_1\right)}{\mathrm{Ai}'(-\lambda_1)}\right|^2. \tag{61}$$

The probability density at a height $z_1^{(2)}$ does not exhibit any time-dependent oscillations. We show the probability density as a function of the height z above a mirror (the y axis) and the time t (the x axis) in a superposition of the first and second gravitational states in Fig. 4.

The position of the node in the wave function of the second state is shown in Fig. 4 as a horizontal line, separating the lower and upper rows of periodic maxima and minima in the probability density plot. The position $z_1^{(2)}$ does not depend on the initial populations of the gravitational states, which makes it beneficial for extracting the spatial scale l_0.

The knowledge of the length l_0 and the energy ε_0 scales allows extraction of the inertial m and gravitational M masses of $\bar{\mathrm{H}}$ from the following expressions:

$$m = \frac{\hbar^2}{2\varepsilon_0 l_0^2}, \tag{62}$$

$$M = \frac{\varepsilon_0}{g l_0}. \tag{63}$$

The equality $m = M$ postulated by the Weak Equivalence Principle (WEP) relates ε_0 and l_0 as follows:

$$\varepsilon_0 = \hbar\sqrt{\frac{g}{2l_0}}, \tag{64}$$

whereas the gravitational time scale Eq.(6) can be written as:

$$\tau_0 = \sqrt{\frac{2l_0}{g}}. \tag{65}$$

One can easily recognize in this expression a classical time of fall from the height l_0 in the Earth's gravitational field.

Thus a measurement of the temporal-spatial probability density dependence of $\bar{\mathrm{H}}$ in a superposition of the two lowest gravitational states would provide full information on the gravitational properties of antimatter. The superposition of three (and more) gravitational states could be useful to search for additional (to gravity) interactions with a spatial scale on the order of l_0. For such a purpose, it is useful to study the probability density at the zero position of each Airy function in the superposition of the states. In particular, the nodes of the second and third gravitational states are the following:

$$z_1^{(2)} = (\lambda_2 - \lambda_1) l_0 = 10.27\ \mu\mathrm{m}, \tag{66}$$

$$z_1^{(3)} = (\lambda_3 - \lambda_2) l_0 = 8.41\ \mu\mathrm{m}, \tag{67}$$

$$z_2^{(3)} = z_1^{(2)} + z_1^{(3)} = 18.68\ \mu\mathrm{m}. \tag{68}$$

The probability density of the three-state superposition (ijk) evaluated at the position $z_i^{(k)}$ is equal to the probability density for the two-state superposition (jk):

$$\left|\Phi_{(ijk)}\left(z_n^{(k)},t\right)\right|^2 = \left|\Phi_{(ij)}\left(z_n^{(k)},t\right)\right|^2. \tag{69}$$

This means that the three-state probability density exhibits harmonic time-dependent oscillation with a frequency ω_{ij} at the height of the node $z_n^{(k)}$. Let us mention that the time dependence of $|\Phi_{ijk}(z,t)|^2$ at any height z, except for the mentioned zeros, is not harmonic; it is given by a superposition of three cosine functions with different frequencies, analogous to Eq. (54). This property allows us to extract the zero positions using the probability density. One can see that knowledge of the single-state nodes [Eqs. (67) and (68)] is analogous to knowledge of the transition frequencies. The measurement of one zero position allows us to extract the spatial scale l_0; the measurement of two or more zero positions allows us to constrain hypothetical nonlinear additions to the gravitational potential. We show the probability density as a function of height z above the mirror (the y axis) and the time t (the x axis) for a superposition of the first, second, and third gravitational state in Fig. 5.

V. FEASIBILITY OF EXPERIMENTS ON GRAVITATIONAL STATES OF ANTIHYDROGEN

In this section we study the feasibility of an experiment on the gravitationally bound quantum states of antihydrogen atoms. For that purpose, we compare such experiment with the already performed experiments on the gravitational states of ultra-cold neutrons (UCNs) [17–19]. The UCN gravitational experiments can be used as a benchmark for such a comparison because (1) the neutron mass is nearly equal to the antihydrogen mass, (2) the modifications of the $\bar{\mathrm{H}}$ quantum state energies and wave-functions following from the precise shape of the Casimir-Polder potential are small compared to the solutions for the quantum bouncer (UCNs in the Earth's gravitational field above a perfect mirror are well described by the quantum bouncer model), (3) our estimations indicate that the lifetimes of $\bar{\mathrm{H}}$ in the quantum states (0.1 s) are comparable to or even longer than the time of UCN passage through the mirror-absorber installation, and (4) UCN velocities are comparable to velocities of ultracold $\bar{\mathrm{H}}$ atoms produced in traps [13,14]. We will discuss here mainly the statistical limitations arising from an estimate of the spectra from sources of $\bar{\mathrm{H}}$ atoms that are projected in the near future.

In the simplest configuration, the experimental method for observation of the gravitational states of neutrons consisted of measuring the flux of UCNs passing through a slit between the horizontal mirror and the flat absorber (scatterer) placed above it at a variable height as a function of the slit height (the integral measuring method), or analyzing the spatial UCN density distribution behind the exit of the horizontal bottom mirror (the differential measuring method) using position-sensitive neutron detectors. The slit height can be changed and precisely measured. The absorber acts selectively on the gravitational states; namely, the states with a spatial size $H_n = \lambda_n^0 l_0$ smaller than the absorber height H are weakly affected, while the states with $H_n > H$ are intensively absorbed [24,28,29]. A detailed description of the experimental method, the experimental setup, the results and their relevance can be found, for instance, in Refs. [17–19,30].

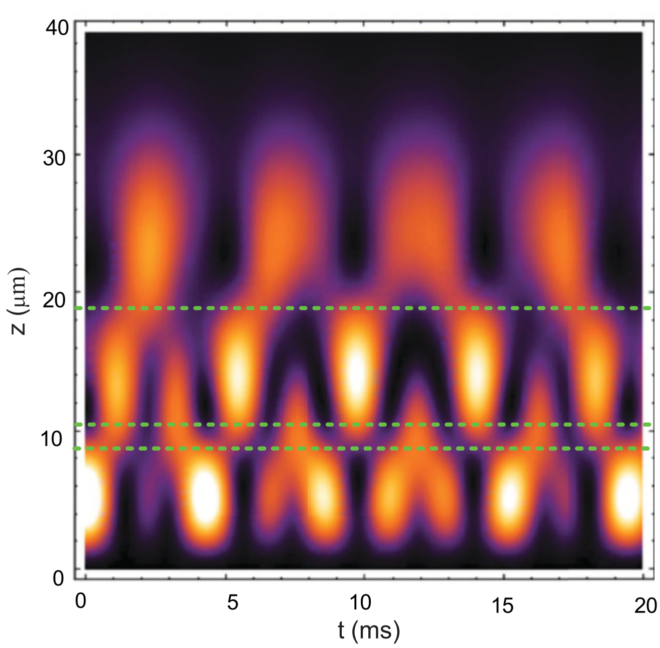

FIG. 5. (Color) The probability density of $\bar{\mathrm{H}}$ in a superposition of the first, second, and third gravitational states, as a function of the height z above the mirror (vertical axis) and the time t (horizontal axis). Dark shade, low probability density; light shade, high probability density. The dashed lines indicate the positions of the nodes in the wave functions of the second and third states.

Leaving aside numerous methodological difficulties in the experiments of this kind (since they have been already overcome in the neutron experiments) and a real challenge to get high phase-space densities of trapped antihydrogen atoms (they are aimed at anyway in the existing antihydrogen projects [10,11]), let us compare relevant phase-space densities in the two problems, keeping in mind that this is the principle parameter, which defines the population of quantum states in accordance with the Liouville theorem. If the phase-space densities of antihydrogen atoms were equal to those of UCNs, we could simply propose using an existing UCN gravitational spectrometer [31,32] for antihydrogen experiments with minor modifications.

UCNs constitute only an extremely narrow initial fraction of a much broader, hotter neutron velocity distribution. Maximum UCN fluxes available today for experiments in a flow-through mode are equal to 4×10^3 UCNs per cm^2/s; such UCNs populate uniformly the phase space up to the so-called critical velocity of about 6 m/s (UCNs with smaller velocity are totally reflected from the surface under any incidence angle; thus they could be stored in closed traps and transported using UCN guides). If one used pulsed mode with a duty cycle of, say, 10^{-3}, the average flux would drop to 4 UCNs per cm^2/s. The pulse method provides more precise measurements, it is used in current experiments with the GRANIT spectrometer, and it will be used in gravitational interference measurements analogous to those performed with the centrifugal quantum states of neutrons [33,34]. Taking into account the phase-space volume available for UCNs in the gravitationally bound quantum states in the GRANIT spectrometer [32], we estimate a total count rate of about 10^2 events/day while the relative accuracy for the gravitational mass is 10^{-3} (we note that the accuracy in the mentioned experiment is defined by the width of a quantum transition, and a few events might be sufficient to observe the corresponding resonance).

The average flux of $\bar{\mathrm{H}}$ atoms projected by the AEGIS collaboration [15,35] is a few atoms per second; let us take it equal to 3 $\bar{\mathrm{H}}$ per second to have it defined. The cloud length is ∼8 mm, its radius is ∼1.5 mm, and thus the cloud volume is $\sim 5 \times 10^{-2}$ cm^3. For comparison, estimates of the average UCN flux, which would be emitted from a small UCN source with a volume of $\sim 5 \times 10^{-2}$ cm^3, the maximum available UCN density of 30 UCNs per cm^3, and a duty cycle of $\sim 10^{-3}$ gives 0.15 UCNs per second. This is 20 times lower than the $\bar{\mathrm{H}}$ flux estimated above. One should not forget, however, that the projected $\bar{\mathrm{H}}$ temperature is ∼100 mK, i.e., 100 times larger than the effective UCN temperature. Thus we lose a factor of $\sqrt{100} = 10$ because of the larger spread of $\bar{\mathrm{H}}$ vertical velocities. No geometrical factors are taken into account here, and no constraints following from final solid angles allowed, final sizes of $\bar{\mathrm{H}}$ detectors, mirrors, etc. However, accounting for these would decrease our estimation by only a few times provided proper experiment design (note that equal acceleration of all antihydrogen atoms would not decrease their phase-space density). Thus we could provide statistical power of an $\bar{\text{H}}$ experiment comparable to a UCN experiment.

Since the projected temperature of antihydrogen atoms in another proposal [36] is significantly lower ($\sim$1 mK, that is, just equal to the effective UCN temperature), the phase-space density of antihydrogen atoms could be even higher. Another significant advantage of a lower temperature consists of a more compact setup design. Note that a gravitational spectrometer analogous to that used in Refs. [31,32] selects just a very small fraction of UCNs ($\bar{\text{H}}$) available (those with extremely small vertical velocity components); thus the count rate of "useful" events is extremely low in both cases.

Thus, we conclude that measurements of the gravitationally bound quantum states of antihydrogen atoms would be realistic if they profited from methodological developments available in neutron experiments and high phase-space densities of antihydrogen atoms targeted by the future experiments. Based on extensive analysis of the mentioned neutron experiments, we can conclude that measurement of the gravitational mass of antihydrogen atoms with an accuracy of at least 10^{-3} is realistic, provided that the projected high $\bar{\text{H}}$ phase-space density is achieved.

VI. CONCLUSIONS

We argue for the existence of long-lived quasistationary states of $\bar{\text{H}}$ above a material surface in the gravitational field of Earth. A typical lifetime of such states above an ideally conducting plane surface is $\tau \simeq 0.1$ s. The quasistationary character of such states is due to the quantum reflection of ultracold (anti)atoms from the Casimir-Polder (anti)atom-surface potential. The relatively long lifetime is due to the smallness of the ratio of the characteristic spatial antiatom-surface interaction scale l_{CP} and the spatial gravitational scale l_0. We show that the spectrum of decaying gravitational levels of $\bar{\text{H}}$ is quasidiscrete even for the highly excited states as long as their quantum number $n \ll 30\,000$.

We argue that low-lying gravitational states provide an interesting tool for studying the gravitational properties of antimatter—in particular, for testing the equivalence between the gravitational and inertial masses of $\bar{\text{H}}$. We show that by counting the number of $\bar{\text{H}}$ annihilation events on the surface, both the transition frequencies between gravitational energy levels and the spatial density distribution of superpositions of gravitational states can be measured. An important observation in this context is that a modification of the above-mentioned properties of gravitational states due to the interaction with a surface disappears in the first order of the small ratio l_{CP}/l_0.

Finally, we show that actual measurements of quantum properties of $\bar{\text{H}}$ atoms levitating above a material mirror in gravitational states are feasible, provided that the projected high phase-space density of $\bar{\text{H}}$ is achieved.

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APPENDIX

Here we derive an expression for the scalar product of the eigenfunctions of two complex-energy gravitational states:

$$\alpha_{ij} = \frac{1}{N_i N_j} \int_0^\infty \text{Ai}^*(z/l_0 - \lambda_j)\,\text{Ai}(z/l_0 - \lambda_i)\,dz, \quad \text{(A1)}$$

with

$$N_i^2 = \int_0^\infty \text{Ai}^2(z/l_0 - \lambda_i)\,dz. \quad \text{(A2)}$$

We start with the equations for the eigenfunction $\text{Ai}(z/l_0 - \lambda_i)$ and the complex eigenvalues $\lambda_i = \lambda_i^0 + a_{\text{CP}}/l_0$:

$$-\text{Ai}''(z/l_0 - \lambda_i) + z\,\text{Ai}(z/l_0 - \lambda_i) = \lambda_i\,\text{Ai}(z/l_0 - \lambda_i). \quad \text{(A3)}$$

The equations for the complex conjugated eigenfunction and the eigenvalue are

$$-\text{Ai}^{*''}(z/l_0 - \lambda_j) + z\,\text{Ai}^*(z/l_0 - \lambda_j) = \lambda_j^*\,\text{Ai}^*(z/l_0 - \lambda_j). \quad \text{(A4)}$$

We multiply both sides of equation Eq. (A3) by $\text{Ai}^*(z/l_0 - \lambda_j)$ and integrate them over z. Then we multiply both sides of Eq. (A4) by $\text{Ai}(z/l_0 - \lambda_i)$ and integrate them over z. After subtraction of the results of these operations we get

$$\begin{aligned} &\text{Ai}^{*'}(-\lambda_j)\,\text{Ai}(-\lambda_i) - \text{Ai}^*(-\lambda_j)\,\text{Ai}'(-\lambda_i) \\ &\quad = (\lambda_j^* - \lambda_i) \int_0^\infty \text{Ai}^*(z/l_0 - \lambda_j)\,\text{Ai}(z/l_0 - \lambda_i)\,dz. \end{aligned} \quad \text{(A5)}$$

To get the above result, we integrated by parts the integrals with second derivatives and took into account that Airy functions vanish at infinity. Now we take into account the equality $\text{Ai}(-\lambda_i^0) = 0$ and smallness of the ratio a_{CP}/l_0 to get the following expressions, exact up to the second order in a_{CP}/l_0:

$$\text{Ai}(-\lambda_i) = -\frac{a_{\text{CP}}}{l_0}\,\text{Ai}'\left(-\lambda_i^0\right), \quad \text{(A6)}$$

$$\text{Ai}^*(-\lambda_j) = -\frac{a_{\text{CP}}^*}{l_0}\,\text{Ai}^{*'}\left(-\lambda_j^0\right). \quad \text{(A7)}$$

Up to the second order in a_{CP}/l_0 we get

$$\begin{aligned} &\text{Ai}^{*'}(-\lambda_j)\,\text{Ai}(-\lambda_i) - \text{Ai}^*(-\lambda_j)\,\text{Ai}'(-\lambda_i) \\ &\quad = i\frac{b}{2l_0}\,\text{Ai}^{*'}\left(-\lambda_j^0\right)\text{Ai}'\left(-\lambda_i^0\right). \end{aligned} \quad \text{(A8)}$$

This result should be combined with the known expression for the normalization coefficient:

$$N_i^2 = \text{Ai}'^2(-\lambda_i) + \lambda_i\,\text{Ai}^2(-\lambda_i), \quad \text{(A9)}$$

which, up to the second order in a_{CP}/l_0, turns to be

$$N_i = \text{Ai}'\left(-\lambda_i^0\right)\left(1 + \lambda_i^0 \frac{a_{\text{CP}}^2}{2l_0^2}\right). \quad \text{(A10)}$$

Keeping first-order terms, we finally get

$$\alpha_{i\neq j} = i\,\frac{b/(2l_0)}{\lambda_j^0 - \lambda_i^0 + ib/(2l_0)}. \quad \text{(A11)}$$

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